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ANISOTROPY OF THE SPECTRUM OF TURBULENCE AT SMALL WAVE-NUMBERS

By G. K. BATCHELOR *and* R. W. STEWART
(*Cavendish Laboratory, Cambridge*)

[Received 29 March 1949]

SUMMARY

In a previous paper it has been shown theoretically that the tendency to isotropic turbulence does not exist for the large-scale components of homogeneous turbulence. This paper provides experimental evidence that the large-scale components of turbulence behind a grid in a uniform stream are anisotropic, even though the total turbulent energy may be distributed with approximate spherical symmetry. The evidence consists (*a*) of a comparison between two different velocity correlations for large spatial intervals, in the initial period of decay, and (*b*) of measurements of the directional distribution of turbulent energy in the final period of decay when all except the large-scale components of the motion have been greatly reduced by viscous decay.

1. Introduction

It is a matter of common observation that there exists a strong tendency to isotropy in all fields of turbulence. When there are no opposing influences such as a non-uniform distribution of mean velocity, isotropic turbulence is quickly set up. This process has been used for many years as a means of producing isotropic turbulence for experimental work in a wind-tunnel. A grid of regularly spaced bars is placed in a uniform stream of air, and at a comparatively short distance downstream from the grid the series of wakes from the various bars has been transformed into isotropic turbulence superimposed on a mean velocity distribution which is once again uniform.

This tendency to spherical symmetry of the turbulence is essentially a consequence of the action of the pressure terms in the Navier–Stokes equations. The static pressure at any point is a non-directional quantity and provides a means of converting energy associated with one component of velocity into energy associated with another component.

Now in another paper (**1**) one of us has investigated the way in which the tendency to isotropy, for a particular Fourier component of the motion, varies with the wave-number of the component in a field of homogeneous turbulence. It was shown there that the energy spectrum, defined as the density in three-dimensional wave-number space of contributions to the energy of the turbulence, is of order k^2 (k = magnitude of wave-number vector) when k is small, and that the rate of change with time of this energy spectrum is of *higher* order in k. When k is small, all three dynamical

effects represented in the Navier–Stokes equations—viscous decay, the tendency to isotropy owing to the pressure term, and the transfer of energy between different wave-numbers owing to the non-linear velocity term—become negligible. Thus the energy of Fourier components of very small wave-number is unchanged throughout the history of the homogeneous turbulence. In particular, this analysis shows that if the homogeneous turbulence is created with an anisotropic distribution of energy for the very small wave-number components of the turbulence, the anisotropy will persist throughout the decay.

In general, anisotropy of the small wave-number components will escape notice in an experiment, because they normally contain a very small proportion of the total energy. However, there are some circumstances in which the permanence of the large-scale components emerges. For instance, the correlation between velocities at two widely separated points is strongly influenced by Fourier components of small wave-number and a lack of isotropy will be made apparent by a comparison of correlations over large space intervals taken in different directions. A lack of isotropy of the large eddies may also be expected to show up clearly in the final period of decay, because then all except the very largest eddies have decayed (**1**). It is the purpose of this note to present experimental evidence to show that a grid of bars in a uniform stream creates homogeneous turbulence in which the components of small wave-number are anisotropic, and remain so throughout the decay as predicted theoretically.

2. The spectrum at small wave-numbers

If the general double-velocity correlation (in a field of homogeneous turbulence) between i- and j-components of the velocity at the points P and P', where $\overrightarrow{PP'} = \mathbf{r}$, be written as $R_i^j(\mathbf{r}), = \overline{u_i u_j'}$, the spectrum tensor is defined as

$$\Gamma_i^j(\mathbf{k}) = \frac{1}{8\pi^3} \iiint R_i^j(\mathbf{r}) e^{-i(\mathbf{k}\cdot\mathbf{r})}\, d\tau(\mathbf{r}), \tag{2.1}$$

where $\mathbf{k}$ is the wave-number vector, $d\tau(\mathbf{r})$ is an element of volume at the position $\mathbf{r}$, and the integration is over the whole of space. The corresponding Fourier transform relation is

$$R_i^j(\mathbf{r}) = \iiint \Gamma_i^j(\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{r})}\, d\tau(\mathbf{k}). \tag{2.2}$$

The condition of incompressibility of the fluid requires $\Gamma_i^j(\mathbf{k})$ to satisfy the equations

$$k_i \Gamma_i^j(\mathbf{k}) = k_j \Gamma_i^j(\mathbf{k}) = 0 \tag{2.3}$$

for all values of $\mathbf{k}$.

It can be shown that, as a consequence of the definition of $\Gamma_i^j(\mathbf{k})$ and of

the continuity condition (2.3), the first non-zero term in the expansion of $\Gamma_i^j(\mathbf{k})$ as a power series† in the components of $\mathbf{k}$ is

$$k_m k_n \Gamma^j_{imn}. \tag{2.4}$$

Here Γ^j_{imn} is a numerical fourth-order tensor which is required by the continuity conditions (2.3) to satisfy the relations

$$\Gamma^j_{imn}+\Gamma^m_{inj}+\Gamma^n_{ijm} = \Gamma^j_{imn}+\Gamma^j_{mni}+\Gamma^j_{nim} = 0. \tag{2.5}$$

The dynamical equation for the rate of change of $\Gamma_i^j(\mathbf{k})$ with time then shows that Γ^j_{imn} is invariant throughout the decay. Hence, as stated already, the initial characteristics of the spectrum at small wave-numbers (e.g. anisotropy) will persist. The term (2.4) normally has very little influence on quantities like the total energy of the spectrum and the consequences of the invariance of Γ^j_{imn} only appear in rather special circumstances, two examples of which are discussed in the next two sections.

3. The velocity correlation at large values of r

We find from (2.1) and (2.4) that

$$\left[\frac{\partial^2\Gamma_i^j(\mathbf{k})}{\partial k_m\,\partial k_n}\right]_{k=0} = 2\Gamma^j_{imn} = -\frac{1}{8\pi^3}\iiint r_m r_n R_i^j(\mathbf{r})\,d\tau(\mathbf{r}). \tag{3.1}$$

This integral is invariant with time and will be isotropic only in the unlikely event that the large eddies are arranged isotropically at the instant of creation of the homogeneous field. The integrand is weighted strongly in favour of large values of r ($= |\mathbf{r}|$), so that any anisotropy of $R_i^j(\mathbf{r})$ which exists may be expected to show up at large values of r, even during the initial period of decay when the turbulent energy is distributed with approximate spherical symmetry.

From an experimental point of view, the best method of testing isotropy of $R_i^j(\mathbf{r})$ is to choose i and j to correspond with velocity fluctuations parallel to the uniform stream in the wind-tunnel, and to vary the direction of $\mathbf{r}$. If x_1 is the downstream coordinate (with origin at the grid), a comparison of $R_1^1(0,r,0)$ and $R_1^1(0,0,r)$ (which should be equal in isotropic turbulence) provides a convenient test of symmetry about the x_1-axis. However, conventional grids consisting of two orthogonal sets of parallel bars are not likely to show very much departure from symmetry about the x_1-axis, even in the large eddies. A grid consisting of *one* set of parallel

† In reference (1) a pure imaginary term of the first degree in k_m was thought to be the first non-zero term of the series. However, since that paper was written J. Kampé de Fériet (2) has derived an (exact) general form for the spectrum tensor, and consistency with this general form demands that the first-degree term in the series should vanish identically.

rods, 0·95 cm. in diameter with a spacing length 2·54 cm., was therefore constructed and the above comparison was made.

With a mean stream velocity of 620 cm./sec. the field was found to become statistically homogeneous at a distance 10 spacing lengths downstream from the grid, at least in so far as the factors U_α/U_β and E_α/E_β both become unity, as shown in Fig. 1; U here denotes the mean stream velocity,

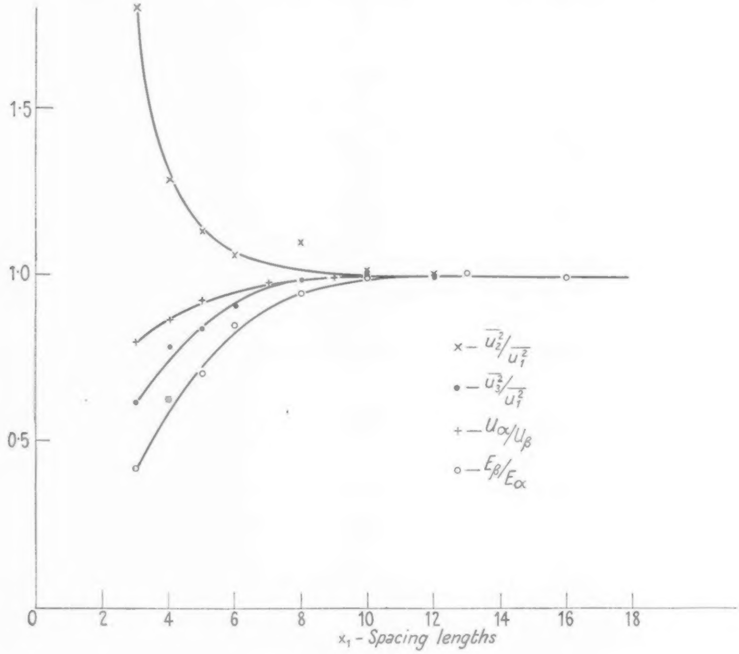

FIG. 1.

E the turbulent energy, and the subscripts α and β refer to the conditions respectively behind and between rods of the grid. The energy of the turbulence is then distributed with approximate spherical symmetry, as shown by the plot of $\overline{u_2^2}/\overline{u_1^2}$ and $\overline{u_3^2}/\overline{u_1^2}$ (where the x_2-axis is parallel to the rods). The isotropy does not, however, extend to the larger scales of motion, and the correlation coefficients

$$R_1^1(0,r,0)/R_1^1(0,0,0) \quad \text{and} \quad R_1^1(0,0,r)/R_1^1(0,0,0)$$

are very appreciably different at large values of r. Fig. 2 shows the comparison of the two correlations at distances of 10, 20, 35, and 60 spacing-lengths downstream. As is to be expected, the correlation between

velocities at two points with a relative displacement parallel to the rods is greater, at large values of r, than that for a relative displacement across the rods.

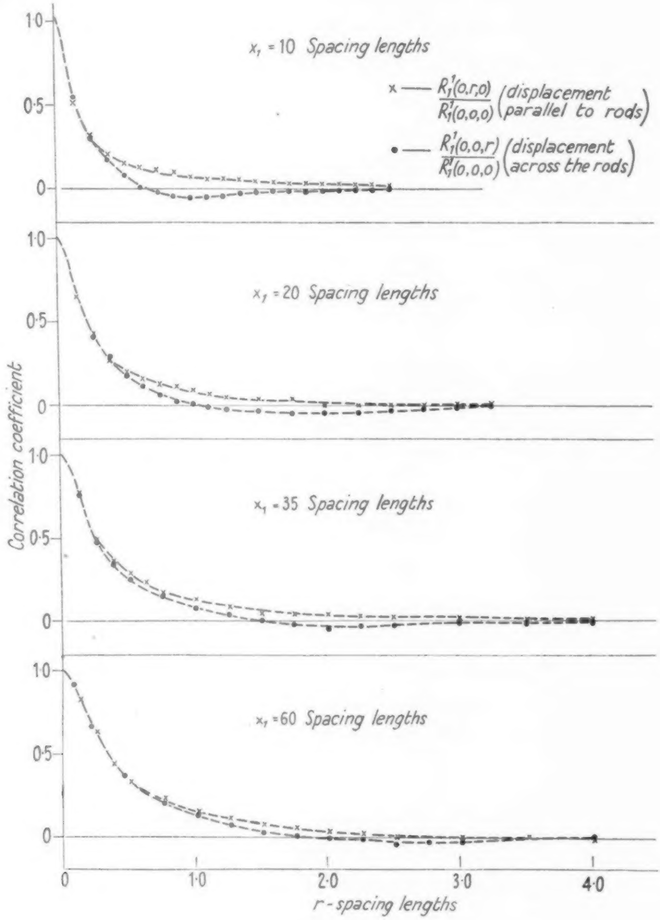

Fig. 2.

4. The final period of decay

The second method used in this paper to exhibit the anisotropy of the small wave-number components of the motion is to measure the directional distribution of turbulent energy in the final period of decay. When the decay reaches an advanced stage, the velocity fluctuations become feeble and the motion is dominated by viscous forces. Under these circumstances

the variation of the spectrum with time is given by (**1**)

$$\Gamma_i^j(\mathbf{k}, t) = \Gamma_i^j(\mathbf{k}, t_0)e^{-2\nu k^2(t-t_0)}. \tag{4.1}$$

When $t-t_0$ is very large, only the behaviour of $\Gamma_i^j(\mathbf{k}, t_0)$ at small values of k is relevant. We have already seen that the expression (2.4) represents the spectrum at small values of k at all times, so that

$$\Gamma_i^j(\mathbf{k}, t) \to k_m k_n \Gamma_{imn}^j e^{-2\nu k^2(t-t_0)} \tag{4.2}$$

as $t-t_0 \to \infty$. In other words, the characteristics of the small wave-number components of the motion at the instant of creation of the homogeneous field eventually determine almost the *whole* of the spectrum.

The correlation tensor corresponding to the asymptotic spectrum tensor (4.2) is found from (2.2) to be

$$R_i^j(\mathbf{r}) = \frac{(\pi^3/2)^{\frac{1}{2}}}{8[\nu(t-t_0)]^{\frac{3}{2}}}\left[\Gamma_{imm}^j - \frac{r_m r_n \Gamma_{imn}^j}{4\nu(t-t_0)}\right]\exp\left[-\frac{r^2}{8\nu(t-t_0)}\right]. \tag{4.3}$$

The mean squares of the three velocity components are obtained by putting $r = 0$;

$$\frac{\overline{u_1^2}}{\Gamma_{1mm}^1} = \frac{\overline{u_2^2}}{\Gamma_{2mm}^2} = \frac{\overline{u_3^2}}{\Gamma_{3mm}^3} = \frac{(\pi^3/2)^{\frac{1}{2}}}{8[\nu(t-t_0)]^{\frac{3}{2}}}. \tag{4.4}$$

The longitudinal correlation coefficient, usually denoted by $f(r)$, is given by

$$f(r) = \frac{R_1^1(r, 0, 0)}{\overline{u_1^2}} = \exp\left[-\frac{r^2}{8\nu(t-t_0)}\right] \tag{4.5}$$

in view of the requirement of the continuity condition (2.5) that $\Gamma_{111}^1 = 0$. Thus the longitudinal correlation coefficient is independent of the direction in which the correlation is taken, and has the form (4.5) whether the turbulence in the final period is isotropic or not.

Experimental confirmation of these predictions about turbulence in the final period of decay behind a conventional grid of two orthogonal rod systems has already been described (**3**). It is shown in that paper that measurements of $\overline{u_1^2}$, where the suffix 1 refers to the direction of the wind-tunnel stream, vary as $(t-t_0)^{-\frac{5}{2}}$, in agreement with (4.4), and that the longitudinal correlation coefficient, measured in the downstream direction, has the shape given by (4.5). However, at the time when these experiments were made, the possibility of the turbulence not being isotropic in the final period after being apparently isotropic earlier in the decay was not appreciated and measurements of $\overline{u_2^2}$ and $\overline{u_3^2}$ were not made.

These measurements have since been undertaken with the same grid. As was pointed out (**3**), the grid used, although appearing regular to the eye, produced appreciable variations in $\overline{u_1^2}$ in the plane normal to the stream at large distances downstream. These spatial variations were also

present in $\overline{u_2^2}$ and $\overline{u_3^2}$. However, the mean of a considerable number of measurements of each of $\overline{u_1^2}$, $\overline{u_2^2}$, and $\overline{u_3^2}$ in the plane normal to the stream varied smoothly with distance downstream, and the averaged results are shown in Fig. 3.

The ratios $\overline{u_1^2}/\overline{u_2^2}$ and $\overline{u_1^2}/\overline{u_3^2}$ (which are equal within experimental error), after being close to unity in the initial period of decay, increase to about

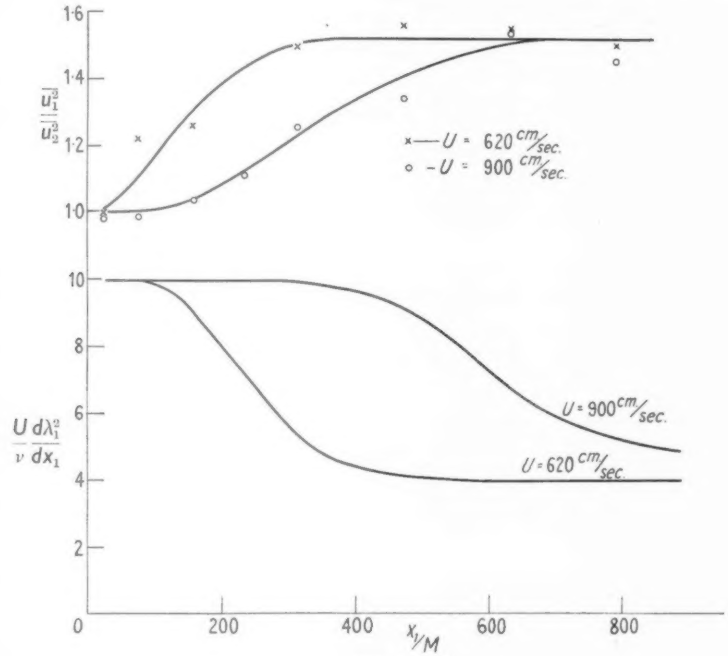

Fig. 3.

1·5 far downstream. Also shown in the figure are calculations of $d\lambda_1^2/dx_1$, made from measurements of λ_1^2 recorded in reference (**3**), where λ_1 is defined by

$$-\frac{\overline{u_1^2}}{\lambda_1^2} = \left[\frac{\partial^2 R_1^1(r,0,0)}{\partial r^2}\right]_{r=0}. \tag{4.6}$$

The asymptotic value of $(U/\nu)d\lambda_1^2/dx_1$ in the final period of decay (see (4.5)) is 4 and the curves show that this value is attained in the neighbourhood of the position at which $\overline{u_1^2}/\overline{u_2^2}$ approaches 1·5. The figure 1·5 (which may vary between 1·3 and 1·7 at particular positions across the flow) appears to be independent of the grid Reynolds number. Presumably, close behind the grid, the system of rods parallel to the x_2-axis produces large contributions

to $\overline{u_1^2}$ and $\overline{u_3^2}$ while the system parallel to the x_3-axis produces large contributions to $\overline{u_1^2}$ and $\overline{u_2^2}$, the net result being that the contribution to $\overline{u_1^2}$ from the largest eddies is permanently greater than the contributions to $\overline{u_2^2}$ and $\overline{u_3^2}$.

Acknowledgement

One of us (R. W. S.) wishes to thank the Ontario Research Commission for a scholarship held during the course of the work.

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THE EQUILIBRIUM OF THIN ELASTIC SHELLS

By A. E. GREEN *and* W. ZERNA

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[Received 28 June 1949]

SUMMARY

A theory of thin elastic shells is developed as a first approximation without using the usual assumption that the normals to the middle surface of the undeformed shell remain normals in the deformed shell. General coordinates are used and the effect of shearing stresses is included so that the theory for plates as developed by Reissner is included as a special case. Vector and tensor notations are used throughout.

1. Introduction

In recent years a great number of papers have been published about the general theory of thin elastic shells. The methods used in these papers are varied and the theories are based on certain approximations which are often rather confusing and difficult to justify. Also there is some disagreement over the differences in the final forms of the results. For the usual theory of shells the following assumptions are made:

1. The thickness t of the shell is small compared with the other dimensions. Thus, if L is a characteristic length, for example, the smallest radius of curvature of the middle surface of the shell, then†

$$\lambda = t/L \ll 1. \tag{1.1}$$

2. The displacements are sufficiently small for quantities of the second and higher orders to be neglected compared with those of the first order.
3. The stress components normal to the middle surface are small compared with the other stress components and may be neglected in the stress-strain relations.
4. The normals of the undeformed middle surface become normals of the deformed middle surface. Sometimes the further assumption is made that the normals suffer no extension.

The first assumption defines what is meant by the term 'thin shell' and may be regarded as satisfactory for many problems. The second assumption is always made in the classical theory of small displacements and small strains in elasticity and it ensures that the differential equations

† λ is not to be confused with one of the elastic constants of Lamé, which is not used here.

remain linear. The third assumption may be reasonable, but the third and fourth assumptions taken together are much less satisfactory and are difficult to justify in a consistent way. Also it is difficult to estimate the effect of these assumptions on the orders of magnitude of certain terms which some writers retain and some reject. These points have been discussed in some papers, but no satisfactory conclusion has yet been reached.

Using the above assumptions, general theories dealing with the equilibrium of thin shells have been developed by Love (**1**), Trefftz (**2**), Reissner (**3**), Reutter (**4**), Byrne (**5**), and Fadle (**6**) in which the lines of curvature of the middle surface are used as coordinates. Theories which are based on general coordinates are developed by Goldenweiser (**7**), Rabotnov (**8**), and Zerna (**9**). It is beyond the purpose of this short review to discuss all these papers in detail. They differ mainly in the methods used for the derivation of the fundamental equations and in the retention or neglect of certain terms which are connected with the initial assumptions. For example, Love (**1**), Reissner (**3**), and Goldenweiser (**7**) neglect certain terms, while Byrne (**5**) and Zerna (**9**) keep these terms and Byrne emphasizes that they should be kept.

Papers which deserve special mention are those by Chien (**10, 11**, **12**) and also a paper by Synge and Chien (**13**), in which, for the first time, a theory is developed which does not use the assumptions made by other writers. Chien's theory seems to be remarkable for its complete generality. All quantities are developed in infinite series in terms of a coordinate x_3 through the thickness of the shell, and it is shown that it is theoretically possible to determine all the coefficients of the series. For practical calculations, however, it is necessary to approximate by considering only a finite number of terms of the series. It is here that some doubts must be expressed about certain of the approximations made by Chien. He appears to assume that the order of magnitudes of the various quantities can be assessed by expanding them as power series in λ, and that differentiation with respect to coordinates in the surface of the shell does not alter the order of magnitude. Simple examples, however, show that these assumptions do not hold in many cases, for some of the quantities in question may depend on functions of the type $\exp(\pm x/\lambda^n)$, where x is a coordinate in the surface of the shell; these functions cannot be expanded in a power series in λ and their orders of magnitude are altered when they are differentiated with respect to x.

It does, in fact, appear to be difficult to decide how many terms in the series expansions in powers of x_3 should be retained. Also the order of the final differential equations which are obtained depends on the number of

terms retained in the power series, and this in turn affects the number of boundary conditions which it is necessary to impose on the edges of the shell. This introduces considerable complications if the order of the differential equations is too high and may make the theory too unwieldy for practical applications.

Again, Chien's theory is developed from the 'intrinsic' point of view so that he avoids using the displacement vector. Although this is theoretically satisfactory it has some disadvantages in practical applications where boundary conditions are often expressed in terms of displacements.

It appears that it is still desirable to attempt to develop a theory of thin shells which can be used for practical applications and which is, as far as possible, a systematic first approximation. At this point some guidance about the method to be used can be obtained from the theory of thin plates, since any theory of shells should contain the plate theory as a special case and since the plate theory is based on a reasonably consistent set of approximations.

A theory of bending of thin plates which includes the effect of transverse shear deformations has been obtained by Reissner (**14**, **15**, **16**) by using Castigliano's theorem of minimum energy, and the results are expressed in terms of certain weighted averages of the displacements and stresses. A similar method can be applied to the stretching of a plate. Green (**17**) has shown that some of the algebraic complications can be avoided, and the same results obtained, by using directly the stress-strain relations and the stress equations of equilibrium. It appears that some similar method to that used by Green for plates should be the most suitable to apply to shell theory, and such a method is developed in the present paper. There are, of course, additional difficulties in the theory of shells owing to the geometrical quantities which are involved. As well as using the methods of approximation which are similar to those used in plate theory, all terms of order λ, or of higher order, are systematically neglected when compared with unity. Attention is confined to shells of constant thickness so that λ is constant, but, apart from the additional algebra which is involved, there is no difficulty in extending the theory to shells of variable thickness provided that the derivatives of λ with respect to coordinates in the middle surface of the shell are of the same order as λ.

The theory is developed with the help of vector and tensor calculus. Small Greek letters used as indices take the values 1, 2, 3 and small Latin indices take the values 1, 2. Unless otherwise stated the usual summation convention holds, so that summation over either of these ranges is signified by the repetition of an index. If the same index appears more than twice

in one term of an equation it is not summed. Partial differentiation is indicated by a comma and covariant differentiation of tensor quantities is indicated by a vertical line. Thus, for example,

$$a_{\alpha,\beta} = \frac{\partial a_\alpha}{\partial \theta_\beta},$$

$$a_{\alpha|\beta} = a_{\alpha,\beta} - \Gamma^\gamma_{\alpha\beta} a_\gamma,$$

$$a_{\alpha\beta|\gamma} = a_{\alpha\beta,\gamma} - \Gamma^\epsilon_{\alpha\gamma} a_{\epsilon\beta} - \Gamma^\epsilon_{\beta\gamma} a_{\alpha\epsilon},$$

where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of the second kind and θ_α are independent parameters. The suffix in θ_α does not indicate that these parameters are covariant since, in general, they have neither covariant nor contravariant properties. The differentials $d\theta^\alpha$, however, are contravariant, and this is indicated by using the upper index. Also the Kronecker delta has the usual meaning

$$\begin{aligned} \delta^i_k &= 1 \quad (i = k,\ k \text{ not summed}), \\ &= 0 \quad (i \neq k). \end{aligned}$$

2. Geometrical relations

The points of the undeformed shell may be described by general curvilinear coordinates x_α. The surface which is determined by $x_3 = 0$ is called the middle surface; x_3 is the perpendicular distance of a point from the middle surface, and if t is the thickness of the shell

$$-\tfrac{1}{2}t \leqslant x_3 \leqslant +\tfrac{1}{2}t. \tag{2.1}$$

The faces of the shell are determined by the equations $x_3 = \pm\frac{1}{2}t$. The parameters x_i represent any general coordinate system on the middle surface.

Dimensionless coordinates may be introduced by putting

$$\theta_i = \frac{x_i}{L_i}, \qquad \theta_3 = \frac{x_3}{t} \quad (i \text{ not summed}),$$

where L_i are constant characteristic lengths which are chosen in such a way that

$$c_i \leqslant \theta_i \leqslant d_i,$$

where the interval (c_i, d_i) is of unit length. Also, from (2.1) it is seen that θ_3 lies between $\pm\frac{1}{2}$.

The position vector of a point of the shell may be denoted by

$$\mathbf{R}^* = \mathbf{r}^* + x_3 \mathbf{e}_3,$$

where $\mathbf{r}^*$ is the position vector of a point of the middle surface and $\mathbf{e}_3$ is

the unit normal vector to the middle surface. Non-dimensional vectors $\mathbf{R}$, $\mathbf{r}$ are now defined by the equations

$$\mathbf{R} = \frac{\mathbf{R}^*}{L}, \qquad \mathbf{r} = \frac{\mathbf{r}^*}{L},$$

where L is a characteristic length which is the minimum radius of curvature of the middle surface, or the minimum of the lengths L_i, whichever is the smaller. The points of the shell are now determined in non-dimensional form by

$$\mathbf{R} = \mathbf{r}+\lambda\theta_3\,\mathbf{e}_3. \tag{2.2}$$

In the rest of the paper only $\mathbf{R}$ and $\mathbf{r}$ will be used, and these may be called the reduced position vectors.

The expression (2.2) is the vectorial representation of a three-dimensional manifold with general coordinates θ_i, θ_3. The covariant base vectors are given by

$$\mathbf{a}_i = \mathbf{R}_{,i} = (\delta_i^r - \lambda\theta_3\,b_i^r)\mathbf{e}_r, \qquad \mathbf{a}_3 = \mathbf{R}_{,3} = \lambda\mathbf{e}_3, \tag{2.3}$$

where $\mathbf{e}_i = \mathbf{r}_{,i}$ are the covariant base vectors of the middle surface, b_i^r are the mixed components of the curvature tensor which is determined by the coefficients b_{ik} of the second fundamental form of the middle surface. Latin indices are raised or lowered with the help of g^{ik} and g_{ik}, the contravariant and covariant components respectively of the metric tensor of the middle surface. These are connected by the relations

$$g_{11} = \mathbf{e}_1\cdot\mathbf{e}_1 = gg^{22}, \qquad g_{22} = \mathbf{e}_2\cdot\mathbf{e}_2 = gg^{11},$$
$$g_{12} = \mathbf{e}_1\cdot\mathbf{e}_2 = -gg^{12}, \qquad g_{il}\,g^{ik} = \delta_l^k,$$

where

$$g = |g_{ik}| = g_{11}\,g_{22} - g_{12}\,g_{12}.$$

In addition

$$\left.\begin{array}{ll|l} \mathbf{e}^k = g^{rk}\mathbf{e}_r, & \mathbf{e}_k = g_{rk}\,\mathbf{e}^r, & \mathbf{e}_{i,k} = \Gamma_{ik}^l\,\mathbf{e}_l + b_{ik}\,\mathbf{e}_3, \\ \mathbf{e}_r\cdot\mathbf{e}^k = \delta_r^k, & \mathbf{e}_3\cdot\mathbf{e}_3 = 1, & \mathbf{e}^i_{,k} = -\Gamma_{kl}^i\mathbf{e}^l + b_k^i\,\mathbf{e}_3, \\ b_{ik} = \mathbf{e}_3\cdot\mathbf{e}_{i,k} = -\mathbf{e}_i\cdot\mathbf{e}_{3,k}, & & \Gamma_{ik}^l = \mathbf{e}^l\cdot\mathbf{e}_{i,k} = \mathbf{e}^l\cdot\mathbf{e}_{k,i} \\ & & \qquad = -\mathbf{e}_k\cdot\mathbf{e}^l_{,i} = g^{lr}\Gamma_{r,ik}, \\ \mathbf{e}_{3,i} = -b_i^l\,\mathbf{e}_l, & \mathbf{e}_{3,i}\cdot\mathbf{e}_{3,k} = b_{il}\,b_k^l, & \Gamma_{r,ik} = \mathbf{e}_r\cdot\mathbf{e}_{i,k} \\ & & \qquad = \tfrac{1}{2}(g_{ri,k}+g_{rk,i}-g_{ik,r}). \end{array}\right\} \tag{2.4}$$

The covariant components of the metric tensor of the manifold (2.2) are

$$\left.\begin{aligned} m_{ik} &= \mathbf{a}_i\cdot\mathbf{a}_k = g_{ik} - 2\lambda\theta_3\,b_{ik} + \lambda^2\theta_3^2\,b_k^l\,b_{il}, \\ m_{i3} &= \mathbf{a}_i\cdot\mathbf{a}_3 = 0, \\ m_{33} &= \mathbf{a}_3\cdot\mathbf{a}_3 = \lambda^2. \end{aligned}\right\} \tag{2.5}$$

If the determinant of the matrix $m_{\alpha\beta}$ is denoted by m, then $\sqrt{m} = \sqrt{(g)}m^*$, where

$$m^* = \lambda(1 - 2\lambda\theta_3\,H + \lambda^2\theta_3^2\,K), \tag{2.6}$$

and where $H = \frac{1}{2}b_r^r$ is the mean curvature, $K = b_1^1\,b_2^2 - b_2^1\,b_1^2$ is the Gaussian

curvature of the middle surface. The contravariant components of the metric tensor are found to be

$$mm^{11} = m_{22}m_{33}, \qquad mm^{12} = -m_{12}m_{33}, \qquad mm^{22} = m_{11}m_{33},$$
$$m^{i3} = 0, \qquad \lambda^2 m^{33} = 1, \tag{2.7}$$

and m^{ik} may be represented by infinite power series in $\lambda\theta_3$, the first terms of which have the form

$$m^{ik} = g^{ik} + 2\lambda\theta_3 b^{ik} + \dots. \tag{2.8}$$

3. General state of stress

The stress across the surface θ_α = constant is defined by stress vectors $\mathbf{p}_\alpha$ and these may be expressed in terms of the contravariant stress tensor $s^{\alpha\beta}$ by the relations

$$\mathbf{p}_\alpha = \frac{s^{\alpha\beta}}{\sqrt{m^{\alpha\alpha}}}\mathbf{a}_\beta. \tag{3.1}$$

Using (2.3) these become

$$\mathbf{p}_\alpha = \frac{1}{\sqrt{m^{\alpha\alpha}}}(\sigma^{\alpha k}\mathbf{e}_k + \lambda s^{\alpha 3}\mathbf{e}_3), \tag{3.2}$$

where
$$\sigma^{\alpha k} = (\delta^k_r - \lambda\theta_3 b^k_r)s^{\alpha r}. \tag{3.3}$$

If
$$\bar{\mathbf{p}}_\alpha = \sqrt{(mm^{\alpha\alpha})}\mathbf{p}_\alpha \tag{3.4}$$

the equations of equilibrium for an infinitesimal volume element cut out of the shell take the simple form

$$\bar{\mathbf{p}}_{\alpha,\alpha} = 0 \tag{3.5}$$

when volume forces are neglected.

When discussing the equilibrium of thin shells it is more convenient to consider stress-resultants and stress-couples through the thickness of the shell, and the three-dimensional equations (3.5) are reduced to a two-dimensional form. Thus, firstly, equations (3.5) are integrated with respect to θ_3 through the thickness of the shell and the resulting equations are expressed in terms of the components of the vectors with the help of the formulae (3.2) and the relation

$$\Gamma^r_{ri} = \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial\theta_i}.$$

This process gives

$$\left.\begin{aligned} N^{rk}|_r - \lambda b^k_r Q^r - \lambda b^k_r R^r + P^k &= 0, \\ N^{rl}b_{rl} + \lambda Q^r|_r + \lambda P^3 &= 0, \end{aligned}\right\} \tag{3.6}$$

where the covariant differentiation means

$$\left.\begin{aligned} N^{rk}|_r &= N^{rk}_{,r} + \Gamma^k_{rl}N^{rl} + \Gamma^r_{rl}N^{kl}, \\ Q^r|_r &= Q^r_{,r} + \Gamma^r_{rl}Q^l. \end{aligned}\right\} \tag{3.7}$$

The quantities N^{ik}, Q^i are given by the following integrals:

$$N^{ik} = \int_{-\frac{1}{2}}^{+\frac{1}{2}} m^*\sigma^{ik}\,d\theta^3, \tag{3.8}$$

$$Q^{i} = \int_{-\frac{1}{2}}^{+\frac{1}{2}} m^* s^{3i}\,d\theta^3, \tag{3.9}$$

and P^α, R^α are determined by the loads on the faces of the shell so that

$$P^\alpha = \{m^* s^{3\alpha}\}_{-\frac{1}{2}}^{+\frac{1}{2}}, \qquad R^\alpha = \{m^*\theta_3 s^{3\alpha}\}_{-\frac{1}{2}}^{+\frac{1}{2}}. \tag{3.10}$$

To obtain equations for the stress-couples, (3.5) is now multiplied by θ_3 and integrated through the thickness of the shell. This gives

$$M^{rk}|_r - Q^k - \tfrac{1}{4}\lambda b_r^k P^r + R^k = 0, \tag{3.11 a}$$

$$M^{rl} b_{rl} + \lambda T^r|_r - \lambda Q^3 + \lambda R^3 = 0, \tag{3.11 b}$$

where

$$\left.\begin{aligned} M^{rk}|_r &= M^{rk}_{,r} + \Gamma^k_{rl} M^{rl} + \Gamma^r_{rl} M^{kl}, \\ T^r|_r &= T^r_{,r} + \Gamma^r_{rl} T^l, \end{aligned}\right\} \tag{3.12}$$

and

$$M^{ik} = \int_{-\frac{1}{2}}^{+\frac{1}{2}} m^*\theta_3\,\sigma^{ik}\,d\theta^3, \tag{3.13}$$

$$T^{i} = \int_{-\frac{1}{2}}^{+\frac{1}{2}} m^*\theta_3\,s^{3i}\,d\theta^3, \tag{3.14}$$

$$Q^{3} = \int_{-\frac{1}{2}}^{+\frac{1}{2}} m^* s^{33}\,d\theta^3. \tag{3.15}$$

The Christoffel symbols in (3.7) and (3.12) are to be calculated from the metric of the middle surface, using the last formulae in (2.4).

Equations (3.6) represent the conditions of equilibrium of the stress-resultants and equations (3.11) give the conditions of equilibrium of the stress-couples. The above derivation of these equations appears to be more straightforward than that which has been given by other writers.

The quantities defined in (3.8) to (3.10) and (3.13) to (3.15) are of a purely mathematical character. It is not difficult, however, to interpret most of them physically.† N^{ik} are related to physical stress-resultants and M^{ik} are related to the physical stress-couples; each forms a contravariant tensor of the second order. Q^i correspond to the shearing forces and form a contravariant vector. P^α and R^i are related to the components of the loads and the moments of the loads about axes in the middle surface.

† See e.g. W. Zerna (**9**).

4. General state of deformation and stress-strain relations

The position vector of the deformed shell may be denoted by $\bar{\mathbf{R}}^* = L\bar{\mathbf{R}}$ and the displacement vector by $\mathbf{V}^* = L\mathbf{V}$. The points of the deformed shell are now defined by the non-dimensional vectors

$$\bar{\mathbf{R}} = \mathbf{R}+\mathbf{V}.$$

The covariant base vectors of the deformed shell are given by

$$\bar{\mathbf{a}}_\alpha = \mathbf{a}_\alpha+\mathbf{V}_{,\alpha},$$

and the covariant components of the metric tensor of the deformed shell are

$$\left.\begin{aligned}\bar{m}_{ik} &= m_{ik}+\mathbf{a}_i\cdot\mathbf{V}_{,k}+\mathbf{a}_k\cdot\mathbf{V}_{,i}\\ \bar{m}_{i3} &= \lambda\mathbf{e}_3\cdot\mathbf{V}_{,i}+\mathbf{a}_i\cdot\mathbf{V}_{,3},\\ \bar{m}_{33} &= m_{33}+2\lambda\mathbf{e}_3\cdot\mathbf{V}_{,3}.\end{aligned}\right\} \tag{4.1}$$

The covariant components of the strain tensor are now found to be

$$e_{\alpha\beta} = \tfrac{1}{2}(\bar{m}_{\alpha\beta}-m_{\alpha\beta}), \tag{4.2}$$

and using (4.1) these become

$$\left.\begin{aligned}e_{ik} &= \tfrac{1}{2}(\mathbf{a}_i\cdot\mathbf{V}_{,k}+\mathbf{a}_k\cdot\mathbf{V}_{,i}),\\ e_{i3} &= \tfrac{1}{2}(\lambda\mathbf{e}_3\cdot\mathbf{V}_{,i}+\mathbf{a}_i\cdot\mathbf{V}_{,3}),\\ e_{33} &= \lambda\mathbf{e}_3\cdot\mathbf{V}_{,3}.\end{aligned}\right\} \tag{4.3}$$

Assuming an isotropic homogeneous medium the stress-strain relations can be put in the form

$$\left.\begin{aligned}(1-2\eta)s^{ik} &= \mu[(1-2\eta)(m^{li}m^{kr}+m^{kl}m^{ir})+2\eta m^{ik}m^{rl}]e_{rl}+2\mu\eta m^{ik}m^{33}e_{33},\\ (1-2\eta)s^{33} &= 2\mu m^{33}[(1-\eta)m^{33}e_{33}+\eta m^{rl}e_{rl}],\\ s^{3k} &= 2\mu m^{33}m^{kl}e_{l3},\end{aligned}\right\} \tag{4.4}$$

where μ is the shear modulus and η is Poisson's ratio.

5. Approximation

Up to now all equations are exact in the sense of the classical theory of elasticity. No assumptions about the thickness of the shell and the state of deformation have been made (except, of course, that the deformations are small). The exact problem of the shell is governed by the equations of equilibrium (3.5), the stress-strain relations (4.4) which can be expressed in terms of the displacements with the help of (4.3), and by the boundary conditions on the faces and edges of the shell. In this form the problem, however, is for most practical cases too complicated and approximations must be introduced.

In developing the theory which is given here the following points have been kept in view:

1. The theory is required to be a first approximation and regards stress-couples and stress-resultants as being of comparable importance. (This means that the approximation is beyond that of the so-called 'membrane' theory in which stress-couples are neglected.)

2. As far as possible the approximate theory should be developed from a consistent set of assumptions. The assumptions made here are

(*a*) the equations of equilibrium (3.5) will not be satisfied exactly but will be replaced by the equilibrium equations (3.6) and (3.11) for the stress-resultants and stress-couples.

(*b*) The thickness of the shell is assumed to be sufficiently small so that (1.1) holds, and all quantities of order λ, or of higher order, are neglected compared with unity.

With assumption (*b*) equations (4.3) become

$$\left.\begin{aligned} e_{ik} &= \tfrac{1}{2}(\mathbf{e}_i\cdot\mathbf{V}_{,k}+\mathbf{e}_k\cdot\mathbf{V}_{,i}),\\ e_{i3} &= \tfrac{1}{2}(\lambda\mathbf{e}_3\cdot\mathbf{V}_{,i}+\mathbf{e}_i\cdot\mathbf{V}_{,3}),\\ e_{33} &= \lambda\mathbf{e}_3\cdot\mathbf{V}_{,3}.\end{aligned}\right\} \tag{5.1}$$

The terms containing λ in (5.1) cannot be neglected since the relative order of magnitude of $\mathbf{V}_{,i}$ and $\mathbf{V}_{,3}$ is unknown.

The displacement vector $\mathbf{V}$ may be expressed in terms of its covariant components by the relation

$$\mathbf{V} = v_r\,\mathbf{e}^r+v_3\,\mathbf{e}^3,$$

and derivatives of $\mathbf{V}$ are given by

$$\left.\begin{aligned} \mathbf{V}_{,i} &= (v_{r|i}-b_{ir}\,v_3)\mathbf{e}^r+(v_{3|i}+b_i^r\,v_r)\mathbf{e}^3,\\ \mathbf{V}_{,3} &= v_{r,3}\,\mathbf{e}^r+v_{3,3}\,\mathbf{e}^3,\end{aligned}\right\} \tag{5.2}$$

where covariant differentiation is to be taken with respect to the middle surface. Thus

$$v_{r|i} = v_{r,i}-\Gamma^l_{ri}\,v_l,$$

while the component v_3, which is perpendicular to the middle surface, is an invariant with respect to transformations of surface coordinates θ_i, and therefore its covariant derivative is equal to its ordinary derivative, i.e. $v_{3|i} = v_{3,i}$. Using (5.2) the strain components (5.1) become

$$\left.\begin{aligned} e_{ik} &= \tfrac{1}{2}(v_{i|k}+v_{k|i}-2b_{ik}\,v_3),\\ e_{i3} &= \tfrac{1}{2}(\lambda v_{3,i}+v_{i,3}+\lambda b_i^r\,v_r),\\ e_{33} &= \lambda v_{3,3}.\end{aligned}\right\} \tag{5.3}$$

Again, using (2.7) and (2.8) and making the above assumption (*b*) about λ, the stress-strain relations (4.4) become

$$\left.\begin{aligned}(1-2\eta)s^{ik} &= \mu[(1-2\eta)(g^{li}g^{kr}+g^{kl}g^{ir})+2\eta g^{ik}g^{rl}]e_{rl}+\frac{2\mu\eta}{\lambda^2}g^{ik}e_{33},\\ \lambda^2(1-2\eta)s^{33} &= \frac{2\mu(1-\eta)}{\lambda^2}e_{33}+2\mu\eta g^{rl}e_{rl},\\ \lambda^2 s^{3k} &= 2\mu g^{kl}e_{l3}.\end{aligned}\right\} \tag{5.4}$$

It is convenient to eliminate e_{33} from the first two sets of equations of (5.4) in order to express the three stresses s^{ik} in terms of the three strains e_{ik} and the stress s^{33}. Thus

$$s^{ik} = \mu E^{iklr}e_{lr}+\lambda^2\frac{\eta}{1-\eta}g^{ik}s^{33}, \tag{5.5}$$

where

$$E^{iklr} = g^{li}g^{kr}+g^{kl}g^{ir}+\frac{2\eta}{1-\eta}g^{ik}g^{lr} \tag{5.6}$$

is a tensor of the fourth order with the symmetric properties

$$E^{iklr} = E^{kilr} = E^{ikrl} = E^{lrik}.$$

The method to be followed in the rest of the paper is suggested by the corresponding method which has been used by Green (**17**) for plates.

Expressions for the stress-resultants N^{ik} and stress-couples M^{ik} are obtained from (5.5) by integrating both sides with respect to θ_3 after multiplication by m^* and $m^*\theta_3$ and using the approximation (*b*) about the order of λ. Hence

$$\left.\begin{aligned}N^{ik}-\lambda^2\frac{\eta}{1-\eta}g^{ik}Q^3 &= \mu\lambda E^{iklr}(V_{l|r}-b_{rl}U_3),\\ M^{ik}-\lambda^2\frac{\eta}{1-\eta}g^{ik}T^3 &= \mu\lambda E^{iklr}(W_{l|r}-b_{rl}W_3),\end{aligned}\right\} \tag{5.7}$$

where

$$V_l = \int_{-\frac{1}{2}}^{+\frac{1}{2}} v_l\,d\theta^3, \qquad W_l = \int_{-\frac{1}{2}}^{+\frac{1}{2}} \theta_3 v_l\,d\theta^3, \tag{5.8}$$

$$U_3 = \int_{-\frac{1}{2}}^{+\frac{1}{2}} v_3\,d\theta^3, \qquad W_3 = \int_{-\frac{1}{2}}^{+\frac{1}{2}} \theta_3 v_3\,d\theta^3, \tag{5.9}$$

$$T^3 = \int_{-\frac{1}{2}}^{+\frac{1}{2}} m^*\theta_3 s^{33}\,d\theta^3. \tag{5.10}$$

The assumption (*b*) used in deriving equations (5.7) implies that

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} v_\alpha\,d\theta^3, \qquad \int_{-\frac{1}{2}}^{+\frac{1}{2}} v_\alpha\theta_3\,d\theta^3, \tag{5.11}$$

are of the same order in λ and that $\int_{-\frac{1}{2}}^{+\frac{1}{2}} v_\alpha \theta_3^n \, d\theta_3$ $(n \geqslant 2)$ is not of greater order. Similarly

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} v_{i,k} \, d\theta^3, \qquad \int_{-\frac{1}{2}}^{+\frac{1}{2}} v_{i,k} \theta_3 \, d\theta^3, \tag{5.12}$$

are of comparable order and $\int_{-\frac{1}{2}}^{+\frac{1}{2}} v_{i,k} \theta_3^n \, d\theta^3$ $(n \geqslant 2)$ is not of greater order. It should be noticed that no assumption is made about the relative orders of magnitude of $\int_{-\frac{1}{2}}^{+\frac{1}{2}} v_i \, d\theta^3$ and $\int_{-\frac{1}{2}}^{+\frac{1}{2}} v_{i,k} \, d\theta^3$.

With the above assumptions only displacement terms of the form (5.8) and (5.9) appear. If powers of λ beyond the first are retained in the geometrical terms further displacement terms must be included. This would, however, take the theory beyond the state of a first approximation and is not required here.†

It is now necessary to make some assumption about the distribution of the shear stresses s^{3i} through the thickness of the shell, and for this purpose a power-series expansion in θ_3 is appropriate. Since the present theory is only concerned with a first approximation it is found, as in the theory of plates, that a quadratic expression in θ_3 must be assumed. If the boundary conditions (3.10) at the faces of the shell $\theta_3 = \pm\frac{1}{2}$ are then used the expression for $m^* s^{3i}$ takes the form

$$m^* s^{3i} = \tfrac{3}{2} Q^i (1-4\theta_3^2) - \tfrac{1}{2} R^i (1-12\theta_3^2) + P^i \theta_3. \tag{5.13}$$

Using (5.13) the quantity T^i defined in (3.14) is

$$T^i = \tfrac{1}{12} P^i. \tag{5.14}$$

The third equation in (5.4) for the shear stresses s^{3i} is now multiplied by $\frac{3}{2} m^*(1-4\theta_3^2)$ and is integrated with respect to θ_3. This process avoids the introduction of unknown values of the displacement at the surfaces of the shell and, with the assumption about λ, gives

$$\lambda(\tfrac{6}{5} Q^i - \tfrac{1}{5} R^i) = \mu g^{il} (\lambda V_{3,l} + 12 W_l). \tag{5.15}$$

where
$$V_3 = \tfrac{3}{2} \int_{-\frac{1}{2}}^{+\frac{1}{2}} (1-4\theta_3^2) v_3 \, d\theta^3, \tag{5.16}$$

the remaining quantities being defined already in (3.9), (3.10), and (5.8).

† Byrne (**5**) retains quantities up to order λ^3 in the geometrical terms but, as Reissner (**18**) and other writers have already mentioned, this does not appear to have much meaning since some terms of order λ have already been neglected in other parts of his theory.

From the second equation in (5.4) it follows in the same way that

$$W_3 = \frac{\lambda^2(1-2\eta)}{24\mu(1-\eta)} S^3 - \frac{\lambda}{8}\frac{\eta}{1-\eta} g^{rl} \int_{-\frac{1}{2}}^{+\frac{1}{2}} (1-4\theta_3^2) v_{r|l}\, d\theta^3 \tag{5.17}$$

where

$$S^3 = \tfrac{3}{2} \int_{-\frac{1}{2}}^{+\frac{1}{2}} m^*(1-4\theta_3^2) s^{33}\, d\theta^3. \tag{5.18}$$

Again, multiplying the second equation in (5.4) by $m^*\theta_3(1-4\theta_3^2)$ and integrating gives, after some simplification,

$$U_3 = V_3 - \frac{\lambda}{2}\frac{\eta}{1-\eta} g^{rl} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \theta_3(1-4\theta_3^2) v_{r|l}\, d\theta^3 + \frac{\lambda^2}{4}\frac{1-2\eta}{\mu(1-\eta)} \bar{S}^3 \tag{5.19}$$

where

$$\bar{S}^3 = \int_{-\frac{1}{2}}^{+\frac{1}{2}} m^*\theta_3(1-4\theta_3^2) s^{33}\, d\theta^3. \tag{5.20}$$

Before reducing the expressions (5.7) for the stress-resultants and stress-couples to their final forms it is necessary to obtain expressions for Q^3 and T^3. Using equations (3.11 b), and (5.14), Q^3 becomes

$$\lambda Q^3 = M^{rl} b_{rl} + \tfrac{1}{12}\lambda P^r|_r + \lambda R^3. \tag{5.21}$$

To obtain a corresponding expression for T^3 the equation of equilibrium (3.5) is multiplied by $\theta_3^2\, \mathbf{e}_3$ and is integrated with respect to θ_3. With (5.13) and the second of equations (3.6) this gives

$$\lambda T^3 = -\tfrac{1}{40} N^{rl} b_{rl} + \tfrac{1}{2} L^{rl} b_{rl} + \tfrac{1}{10}\lambda P^3 + \tfrac{1}{60}\lambda R^r|_r, \tag{5.22}$$

where

$$L^{ik} = \int_{-\frac{1}{2}}^{+\frac{1}{2}} m^*\theta_3^2\, \sigma^{ik}\, d\theta^3.$$

If now (5.17) and (5.19) are substituted in (5.7), and if the assumption about λ is used, the following equations are obtained for N^{ik}, M^{ik}:

$$N^{ik} = \mu\lambda E^{iklr}(V_{r|l} - b_{rl} V_3) + X^{ik}, \tag{5.23}$$

$$M^{ik} = \mu\lambda E^{iklr} W_{r|l} + Y^{ik}, \tag{5.24}$$

where

$$X^{ik} = \frac{\lambda^2\eta}{1-\eta}\left[g^{ik} Q^3 - \frac{\lambda(1-2\eta)}{4\eta} b_{rl} E^{iklr} \bar{S}^3\right],$$

$$Y^{ik} = \frac{\lambda^2\eta}{1-\eta}\left[g^{ik} T^3 - \frac{\lambda(1-2\eta)}{24\eta} b_{rl} E^{iklr} S^3\right].$$

Because of the assumptions which have been made, the first groups of terms on the right-hand sides of (5.23) and (5.24) are of the same order

of magnitude and the quantities X^{ik} and Y^{ik} can, at most, be of this same order, and may be of smaller order. The stress-resultants N^{ik} and stress-couples M^{ik} are therefore of comparable order. Also, the quantities $\bar{S}^3$ and T^3 are, in general, of comparable orders of magnitude, as are also S^3 and Q^3. It can then be shown that, whatever the relative orders of magnitude of T^3 and Q^3, the terms in $\bar{S}^3$ and S^3 in X^{ik} and Y^{ik} respectively may be omitted. Then if λQ^3, λT^3 are substituted from (5.21) and (5.22) in (5.23) and (5.24) respectively, and, if it is remembered that in general, λL^{ik} can be neglected compared with M^{ik}, the stress-resultants and stress-couples can now be expressed in terms of the five unknown quantities V_α, W_i. Thus

$$N^{ik}-\frac{\lambda^2\eta}{1-\eta}g^{ik}(R^3+\tfrac{1}{12}P^r|_r) = \mu\lambda E^{iklr}(V_{r|l}-b_{rl}V_3), \tag{5.25}$$

$$M^{ik}-\frac{\lambda^2\eta}{10(1-\eta)}g^{ik}(P^3+\tfrac{1}{6}R^r|_r) = \mu\lambda E^{iklr}W_{r|l}. \tag{5.26}$$

The shearing forces Q^i have already been expressed in terms of the displacements V_3 and W_i in (5.15) and the expressions for these forces are repeated here for completeness:

$$\tfrac{6}{5}Q^i-\tfrac{1}{5}R^i = \frac{\mu}{\lambda}g^{il}(\lambda V_{3,l}+12W_l). \tag{5.27}$$

The shell problem is now governed by the five differential equations (3.6) and (3.11 a) and the equations (5.25) to (5.27).

In the special case of the flat plate the above theory reduces to that given by Reissner (**14**, **15**, **16**) as far as bending is concerned. It should be noticed that Reissner's theory for the bending of a plate requires that three boundary conditions are to be satisfied at the edges of a plate in contrast to two boundary conditions in the classical bending theory. In addition, the stretching theory for plates requires two boundary conditions, making five in all. Similarly, five boundary conditions are required in the present theory of shells, in contrast to four in the usual theory.

When the shearing forces in (5.27) are neglected, and orthogonal curvilinear coordinates are used, a theory is obtained which does not contain certain terms which are introduced by some other writers, but, as mentioned earlier, the inclusion of these extra terms only appears to be necessary when proceeding beyond a first approximation. The justification for neglecting the shearing forces is not obvious but there may be many examples where this is satisfactory. The inclusion of shearing forces is likely to be of importance mainly for 'edge' effects.

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ON THE PLANE STRESS-DISTRIBUTION IN AN INFINITE PLATE WITH A RIM-STIFFENED ELLIPTICAL OPENING

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SUMMARY

A solution is given for the decrease of stress concentration factor at an elliptical opening in a plate under uniform plane stress at infinity when a compact stiffening rim is applied to the opening. We begin with the known solution for a plain elliptical opening, on to which is superimposed a stress distribution satisfying the boundary conditions between rim and plate and vanishing at infinity, and the distributions of cross-section of the rim corresponding to these are derived. Particular solutions are obtained for hydrostatic tension and pure shear in two directions at infinity, and from these all other cases may be solved. For ratios of major to minor axis which are not too great, and for rims of moderate cross-section, the cross-sections corresponding to the given solutions vary little round the opening.

Symbols

α, β, elliptical coordinates.
a, b, semi-major and semi-minor axes of elliptical opening.
c, semi-interfocal distance for elliptical coordinate system.
$\widehat{\alpha\alpha}$, $\widehat{\beta\beta}$, $\widehat{\alpha\beta}$, stresses in elliptical coordinates.
n, any positive or negative integer.
P_n, Q_n, arbitrary constants.
A, area of rim cross-section at any point.
r, radius of opening at any point.
t, plate thickness.
y, A/rt.
ν, Poisson's ratio.
K, Y, F_j, C, constants.

Previous work

The reduction of stress concentration due to a rim homogeneously attached to the periphery of a circular opening has received some attention in the literature (**1, 2, 3**). The rim is assumed to be of uniform cross-sectional area A, placed compactly at the edge of the opening so that the stress in it is uniformly distributed over the cross-section. If the radius of the opening is r, and the plate thickness is t, then the parameter $y = A/rt$ defines the influence of the rim on the stress concentration factor. If the radial and circumferential stresses in the plate, at a point where the polar coordinates are r, θ, are $\widehat{rr}$, $\widehat{\theta\theta}$ and the shear stress is $\widehat{r\theta}$, from

considerations of strain equality and equilibrium the boundary conditions at the periphery of the opening become

$$\widehat{\theta\theta} = \widehat{rr}(1+\nu y)/y, \qquad \widehat{r\theta} = -d\widehat{rr}/d\theta, \tag{1}$$

where ν is Poisson's ratio. At infinity all combinations of principal stresses may be obtained by superposition of multiples of the solutions embodying hydrostatic tension, for which

$$\widehat{rr} = \widehat{\theta\theta} = 1, \qquad \widehat{r\theta} = 0, \tag{2}$$

and pure shear, for which

$$\widehat{\theta\theta} = \cos 2\theta, \qquad \widehat{rr} = -\cos 2\theta, \qquad \widehat{r\theta} = -\sin 2\theta. \tag{3}$$

The stresses in the plate at the periphery of the opening, and the rim stress $\widehat{\theta\theta}_r$ corresponding to these cases are, for hydrostatic tension at infinity,

$$\begin{aligned} \widehat{\theta\theta} &= 2(1+\nu y)F_j, & \widehat{rr} &= 2yF_j \quad (j=1) \\ \widehat{\theta\theta}_r &= 2F_j, & \widehat{r\theta} &= 0, \end{aligned} \tag{4}$$

where F_j is defined as $1/\{1+y(j+\nu)\}$, and for pure shear at infinity

$$\begin{aligned} \widehat{\theta\theta} &= -4\cos 2\theta(1+\nu y)F_j, & \widehat{rr} &= -4\cos 2\theta\, yF_j \quad (j=3) \\ \widehat{\theta\theta}_r &= -4\cos 2\theta\, F_j, & \widehat{r\theta} &= -8\sin 2\theta\, yF_j. \end{aligned} \tag{5}$$

The circumferential plate stress $\widehat{\theta\theta}$ at the periphery of the opening is always greater than the rim stress $\widehat{\theta\theta}_r$, and the stress concentration factor is determined either by $\widehat{\theta\theta}$ or $\widehat{r\theta}$, which assumes importance when y is large. For hydrostatic tension there is no limit to the economical value of y, but for pure shear the optimum is reached when $y = 1/(4-\nu)$ on the maximum shear criterion, and the stress concentration factor is reduced from 4 to 16/7. All other cases fall between these two extremes.

Boundary conditions: elliptical opening

To maintain continuity of nomenclature the rim parameter for the elliptical opening is defined as A/rt, where r is now the radius of curvature of the opening at each point under consideration (Fig. 1). In the general case neither A nor r will now be constant round the periphery. As before, there is the condition, using the notation adopted by Inglis (**4, 5**),

$$\widehat{\beta\beta} = \widehat{\alpha\alpha}(1+\nu y)/y,$$

which can be restated as
$$y = \frac{\widehat{\alpha\alpha}}{\widehat{\beta\beta}-\nu\widehat{\alpha\alpha}}. \tag{6}$$

For shear equilibrium along the periphery the equation now becomes

$$\frac{d\beta}{ds}\frac{d}{d\beta}(\widehat{\alpha\alpha}r) = -\widehat{\alpha\beta}, \tag{7}$$

where s is the distance measured along the periphery. In elliptical coordinates this is rewritten

$$6\widehat{\alpha\alpha}\sin 2\beta+(\cosh 2\alpha_0-\cos 2\beta)\frac{d\widehat{\alpha\alpha}}{d\beta} = -\widehat{\alpha\beta}\sinh 2\alpha_0. \tag{8}$$

By expressing the equations of equilibrium for plane stress in terms of displacements, and introducing the complex variable, Inglis has obtained general expressions for distributions of stresses $\widehat{\alpha\alpha}$, $\widehat{\beta\beta}$, and $\widehat{\alpha\beta}$ in an

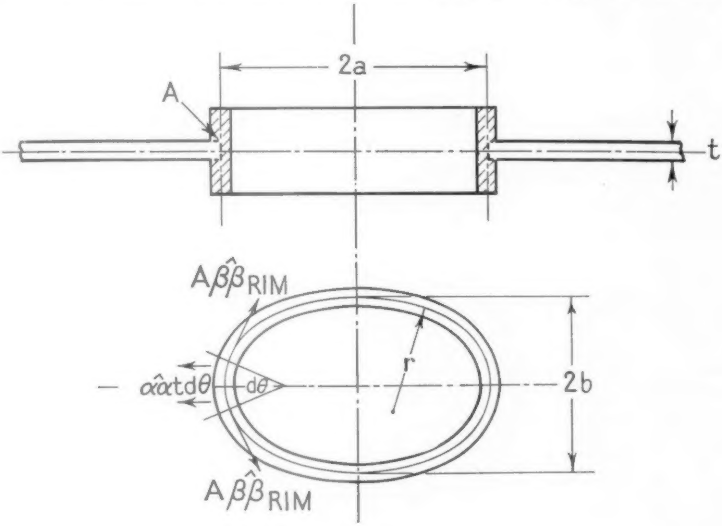

FIG. 1.

elliptical coordinate system in terms of infinite series, the terms of which depend on the integer n and arbitrary constants P_n and Q_n. Substitution of these general expressions into equation (8) yields the following equation in P_n, Q_n, and n:

$$P_n\begin{bmatrix} (n-1)(n-2)[e^{-5\alpha}\sin(n-1)\beta-2e^{-3\alpha}\sin(n-3)\beta+e^{-\alpha}\sin(n-5)\beta] \\ -(n^2-9n+2)[e^{-3\alpha}\sin(n+1)\beta-2e^{-\alpha}\sin(n-1)\beta+e^{\alpha}\sin(n-3)\beta] \\ -(n^2+9n+2)[e^{-\alpha}\sin(n+3)\beta-2e^{\alpha}\sin(n+1)\beta+e^{3\alpha}\sin(n-1)\beta] \\ (n+1)(n+2)[e^{\alpha}\sin(n+5)\beta-2e^{3\alpha}\sin(n+3)\beta+e^{5\alpha}\sin(n+1)\beta] \end{bmatrix}$$

$$+n(n+2)Q_n\begin{bmatrix} e^{-5\alpha}\sin(n+1)\beta-2e^{-3\alpha}\sin(n-1)\beta+e^{-\alpha}\sin(n-3)\beta \\ -2[e^{-3\alpha}\sin(n+3)\beta-2e^{-\alpha}\sin(n+1)\beta+e^{\alpha}\sin(n-1)\beta] \\ e^{-\alpha}\sin(n+5)\beta-2e^{\alpha}\sin(n+3)\beta+e^{3\alpha}\sin(n+1)\beta \end{bmatrix} = 0 \tag{9}$$

again capable of expansion into a series, and the same is true whether the

functions of β in $\widehat{\alpha\alpha}$, $\widehat{\alpha\beta}$ in the general expressions are cos, sin or sin, $-$cos respectively.

At infinity three sets of boundary conditions must be considered, to cover all principal stress combinations. These are

1. *Hydrostatic tension*:

$$\widehat{\alpha\alpha} = \widehat{\beta\beta} = 1, \qquad \widehat{\alpha\beta} = 0.$$

2. *Pure shear*:

$$\widehat{\alpha\alpha} = -\widehat{\beta\beta} = \cos 2\beta, \qquad \widehat{\alpha\beta} = -\sin 2\beta.$$

3. *Pure shear at $\frac{1}{4}\pi$*:

$$\widehat{\alpha\alpha} = -\widehat{\beta\beta} = \sin 2\beta, \qquad \widehat{\alpha\beta} = -\cos 2\beta. \tag{10}$$

Subsidiary stress expressions

Equation (8) is automatically satisfied by the stress expressions representing no rim. Calling these stresses at the periphery $\widehat{\alpha\alpha}_1$, $\widehat{\beta\beta}_1$, $\widehat{\alpha\beta}_1$ ($\widehat{\alpha\alpha}_1 = \widehat{\beta\beta}_1 = 0$), and those of any general expression, vanishing at infinity and satisfying equation (8), $\widehat{\alpha\alpha}_2$, $\widehat{\beta\beta}_2$, $\widehat{\alpha\beta}_2$, by the principle of superposition

$$\widehat{\alpha\alpha} = K\widehat{\alpha\alpha}_2, \qquad \widehat{\beta\beta} = \widehat{\beta\beta}_1 + K\widehat{\beta\beta}_2, \qquad \widehat{\alpha\beta} = K\widehat{\alpha\beta}_2, \tag{11}$$

where K is a constant, represents a further solution of equation (8). Equation (6) for y now becomes

$$y = K\widehat{\alpha\alpha}_2/(\widehat{\beta\beta}_1 + K\widehat{\beta\beta}_2 - \nu K\widehat{\alpha\alpha}_2), \tag{12}$$

from which it is evident that

$$\frac{\widehat{\alpha\alpha}}{y} = \frac{\widehat{\beta\beta}}{1+\nu y} = \frac{\widehat{\alpha\alpha}_2}{\widehat{\alpha\beta}_2}\frac{\widehat{\alpha\beta}}{y} = \widehat{\beta\beta}_1 \Big/ \left\{1 + y\left(-\frac{\widehat{\beta\beta}_2}{\widehat{\alpha\alpha}_2} + \nu\right)\right\}, \tag{13}$$

while for the rim $$\widehat{\beta\beta}_r = \widehat{\beta\beta}_1 \Big/ \left\{1 + y\left(-\frac{\widehat{\beta\beta}_2}{\widehat{\alpha\alpha}_2} + \nu\right)\right\}.$$

Subsidiary stress expressions for superimposition on those representing no rim, vanishing at infinity and satisfying equation (9), must therefore be found, with the additional qualification that a rational value for y in equation (12) be the result.

By elimination of values of n giving finite stresses at infinity and unsuitable distributions of stress at $\alpha = \alpha_0$ in the general stress expressions the number of terms in the series is considerably narrowed. Suitable expressions are obtained by putting $Q_{-1} = 1$ for hydrostatic tension and $P_1 = 1$, $Q_1 = -2e^{2\alpha_0}$ for both cases of pure shear. Substituting these constants into the general stress expressions the following values of the stresses at $\alpha = \alpha_0$ are obtained:

1. *Hydrostatic tension*:

$$\widehat{\alpha\alpha}_2 = -\widehat{\beta\beta}_2 = -2C^2 \sinh 2\alpha_0, \qquad \widehat{\alpha\beta}_2 = -2C^2 \sin 2\beta. \tag{14}$$

2. *Pure shear*:

$$\begin{aligned} -\widehat{\alpha\alpha}_2 \operatorname{cosech} 2\alpha_0 &= \widehat{\beta\beta}_2/[(2\cosh 2\alpha_0+\sinh 2\alpha_0)-2\cos 2\beta] \\ &= 4C^2(e^{-2\alpha_0}-\cos 2\beta), \\ \widehat{\alpha\beta}_2 &= 4C^2 \sin 2\beta(e^{2\alpha_0}-\cos 2\beta). \end{aligned} \tag{15}$$

3. *Pure shear at $\frac{1}{4}\pi$*:

$$\begin{aligned} \widehat{\alpha\alpha}_2 \operatorname{cosech} 2\alpha_0 = -\widehat{\beta\beta}_2/[(2\cosh 2\alpha_0+\sinh 2\alpha_0)-2\cos 2\beta] &= 4C^2 \sin 2\beta, \\ \widehat{\alpha\beta}_2 = 4C^2(1-e^{2\alpha_0}\cos 2\beta)(1-e^{-2\alpha_0}\cos 2\beta)&, \end{aligned} \tag{16}$$

where $C = 1/(\cosh 2\alpha_0-\cos 2\beta)$.

Results

The solutions for $\widehat{\alpha\alpha}_1$, $\widehat{\beta\beta}_1$, $\widehat{\alpha\beta}_1$ are well known, so that the final solutions for $\alpha = \alpha_0$ from equations (13) become

1. *Hydrostatic tension*:

$$\begin{gathered} \frac{\widehat{\alpha\alpha}}{y} = \frac{\widehat{\beta\beta}}{1+\nu y} = 2F_j\, C \sinh 2\alpha_0, \qquad \frac{\widehat{\alpha\beta}}{y} = 2F_j\, C \sin 2\beta, \\ \widehat{\beta\beta}_r = 2F_j\, C \sinh 2\alpha_0 \quad (j = 1). \end{gathered} \tag{17}$$

2. *Pure shear*:

$$\begin{aligned} \frac{\widehat{\alpha\alpha}}{y} &= \frac{\widehat{\beta\beta}}{1+\nu y} = 2F_j\, C(1-e^{2\alpha_0}\cos 2\beta), \\ \frac{\widehat{\alpha\beta}}{y} &= -2F_j\, C\frac{e^{2\alpha_0}\sin 2\beta(e^{2\alpha_0}-\cos 2\beta)}{\sinh 2\alpha_0}, \\ \widehat{\beta\beta}_r &= 2F_j\, C(1-e^{2\alpha_0}\cos 2\beta) \quad (j = m). \end{aligned} \tag{18}$$

3. *Pure shear at $\frac{1}{4}\pi$*:

$$\begin{aligned} \frac{\widehat{\alpha\alpha}}{y} &= \frac{\widehat{\beta\beta}}{1+\nu y} = -2F_j\, Ce^{2\alpha_0}\sin 2\beta, \\ \frac{\widehat{\alpha\beta}}{y} &= -2F_j\, C\frac{e^{2\alpha_0}(1-e^{2\alpha_0}\cos 2\beta)(1-e^{-2\alpha_0}\cos 2\beta)}{\sinh 2\alpha_0}, \\ \widehat{\beta\beta}_r &= -2F_j\, Ce^{2\alpha_0}\sin 2\beta \quad (j = m), \end{aligned} \tag{19}$$

where
$$m = \frac{2\cosh 2\alpha_0+\sinh 2\alpha_0-2\cos 2\beta}{\sinh 2\alpha_0}. \tag{20}$$

These expressions reduce to those for the circular opening when α_0 becomes infinite. If y at $\beta = \frac{1}{4}\pi$ be termed y_O, substituting in equation (12) yields the following relation

$$y = y_O\frac{\widehat{\alpha\alpha}_2\,\widehat{\beta\beta}_{1\cdot 0}}{\widehat{\beta\beta}_1\,\widehat{\alpha\alpha}_{2\cdot 0}+y_O[(\widehat{\beta\beta}_{1\cdot 0}\,\widehat{\beta\beta}_2-\widehat{\beta\beta}_1\,\widehat{\beta\beta}_{2\cdot 0})-\nu(\widehat{\beta\beta}_{1\cdot 0}\,\widehat{\alpha\alpha}_2-\widehat{\beta\beta}_1\,\widehat{\alpha\alpha}_{2\cdot 0})]}, \tag{21}$$

where $\widehat{\alpha\alpha}_{2\cdot 0}$, $\widehat{\beta\beta}_{1\cdot 0}$, $\widehat{\beta\beta}_{2\cdot 0}$ denote their values at $\alpha = \alpha_0$ and $\beta = \frac{1}{4}\pi$. This reduces to

$$y = y_O \frac{\cosh 2\alpha_0}{\cosh 2\alpha_0 - \cos 2\beta\{1 + y_O(1+\nu)\}} \tag{22}$$

for the three cases dealt with. Superposition is therefore straightforward. Again the relation is $y = y_O$ for α_0 equal to infinity.

To determine the stresses in the interior of the plate, K in equation (11) is given by

$$K = \frac{\widehat{\beta\beta}_{1\cdot 0}}{\widehat{\alpha\alpha}_{2\cdot 0}} \frac{y_O}{1 + y_O(-\widehat{\beta\beta}_{2\cdot 0}/\widehat{\alpha\alpha}_{2\cdot 0} + \nu)}. \tag{23}$$

At $\alpha = \alpha_0$ maximum values of $\widehat{\alpha\beta}$ for cases 1 and 2, and $\widehat{\beta\beta}$ for case 3, occur away from the major and minor axes of the opening. By substituting y_O for y and differentiating with respect to β, the values of $\tan\beta$ at which the maxima occur are given by

1. *Hydrostatic tension* ($\widehat{\alpha\beta}$):

$$\tan^2\beta = \frac{\sqrt{(\cosh^2 2\alpha_0 + 8)} - 3}{\cosh 2\alpha_0 + 1}.$$

2. *Pure shear* ($\widehat{\alpha\beta}$):

$$\tan^2\beta = \frac{\sinh 2\alpha_0[\sqrt{(9 + 2e^{2\alpha_0}\sinh 2\alpha_0)} - 3]}{(\cosh 2\alpha_0 + 1)(e^{2\alpha_0} + 1)}.$$

3. *Pure shear at* $\frac{1}{4}\pi$ ($\widehat{\beta\beta}$):

$$\tan^2\beta = \frac{\cosh 2\alpha_0[\sqrt{\{9(1-Y)^2 + \tanh^2 2\alpha_0(Y^2\cosh^2 2\alpha_0 - 1)\}} + 3(1-Y)]}{(\cosh 2\alpha_0 + 1)(Y\cosh 2\alpha_0 + 1)},$$

where

$$Y = \frac{1 + \nu y_O}{1 + y_O(1+\nu)}. \tag{24}$$

When α_0 becomes infinite these expressions give $\tan\beta = \pm 1$, which agrees with the result for the circular opening. The remaining stresses at the periphery have their maxima at $\beta = 0$ or $\frac{1}{2}\pi$. It is important to note that the stresses to which equations (24) refer may be those determining the stress concentration factor at the opening if the amount of rim stiffening is large.

Equation (22) gives different distributions of y on the periphery of the opening for the various values, both of b/a and y_O. The area of cross-section, A, given by yrt, may be expressed as a fraction of A_O, that at $\frac{1}{4}\pi$, and this fraction has been plotted against β for $b/a = 2/3$ and $1/2$ with various values of y_O in Fig. 2. When y_O is small A tends to be larger at the minor axis than at the major, but this tendency is reversed as y_O increases.

Application of results

The solutions assume distributions of rim cross-section at the periphery which vary slightly and may not therefore be wholly practicable. It would seem reasonable, however, to invoke St. Venant's principle, by applying the solutions to isolated points on the periphery, using the values of y

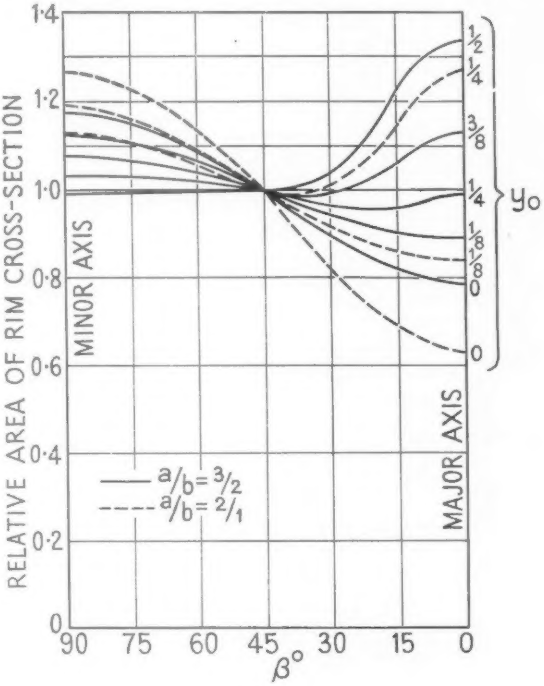

FIG. 2.

dictated by the design at these points. For this purpose equations (17), (18), and (19) would be used as they stand.

Given the values A, a, b, and t, y may be calculated as follows:

$$y = \frac{A\sqrt{2}\sinh 2\alpha_0}{ct(\cosh 2\alpha_0 - \cos 2\beta)^{\frac{3}{2}}}. \tag{25}$$

If $k = b/a$, stress values at the periphery at the major and minor axes may be simplified and become:

At the major axis

$$m = k\left(2+\frac{1}{k}\right), \qquad y = \frac{1}{k}\frac{A}{bt}.$$

At the minor axis

$$m = \frac{1}{k}(2+k), \qquad y = k\frac{A}{at}. \tag{26}$$

1. *For hydrostatic tension.*

At the major axis

$$\widehat{\beta\beta} = \frac{2(1+\nu y)}{k\{1+y(1+\nu)\}}.$$

At the minor axis

$$\widehat{\beta\beta} = \frac{2k(1+\nu y)}{1+y(1+\nu)}. \tag{27}$$

2. *For pure shear.*

At the major axis

$$\widehat{\beta\beta} = -2\left(1+\frac{1}{k}\right)\frac{1+\nu y}{1+y(m+\nu)}.$$

At the minor axis

$$\widehat{\beta\beta} = \frac{2(1+k)(1+\nu y)}{1+y(m+\nu)}. \tag{28}$$

3. *For pure shear at $\frac{1}{4}\pi$.*

At the major axis

$$\widehat{\alpha\beta} = \frac{2y(1+k)^2}{k\{1+y(2k+1+\nu)\}}.$$

At the minor axis

$$\widehat{\alpha\beta} = \frac{-2y(1+k)^2}{k+y\{2+k(1+\nu)\}}, \tag{29}$$

and for the distribution of y round the opening specified by the solutions

At the major axis

$$\frac{A}{A_0} = \left(\frac{2}{1+k^2}\right)^{\frac{1}{2}}\frac{2k^3}{2k^2-(1-k^2)y_O(1+\nu)}.$$

At the minor axis

$$\frac{A}{A_0} = \left(\frac{2}{1+k^2}\right)^{\frac{1}{2}}\frac{2}{2+(1-k^2)y_O(1+\nu)}, \tag{30}$$

where

$$A_0 = \frac{y_O\,at}{k\{2/(1+k^2)\}^{\frac{3}{2}}}.$$

Conclusions

For an elliptical opening stiffened by means of a homogeneous rim an exact and rational solution is obtained when the area of cross-section of the rim follows a distribution round the opening which depends both on the eccentricity of the ellipse and the amount of stiffening. For values of these variables to be expected in practice the distribution does not depart far from uniformity. In practice also, calculations may be carried out for

any particular point by use of the value of rim parameter there and invoking St. Venant's principle. By this means it is probable that the solutions presented may be applied, with only a nominal loss of accuracy, to elliptical openings having a uniform cross-section of rim.

Except for the case of hydrostatic tension there is a maximum limit to the economical cross-sectional area of stiffening which may be applied, since reduction of the stress in the plate tangential to the opening is only achieved at the expense of boundary shear stress in the plate, necessary to build up load in the rim. This limit depends on the state of stress in the plate at infinity, and is determined qualitatively by equating maximum tangential and twice the shear stresses at the periphery of the opening.

For the particular case of the circular opening all the solutions presented revert to those obtained by Timoshenko, Beskin, and others.

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ANALYSIS OF PIN-JOINTED REDUNDANT PLANE FRAMEWORKS USING EQUIVALENT ELASTIC SYSTEMS

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SUMMARY

By introducing the concept of equivalent elastic systems, it is shown how the forces in the members of multiply-redundant pin-jointed plane frameworks may be obtained directly as the result of simple arithmetical operations only. The method of analysis proposed is divided into two distinct parts of which one is an expression of the elastic properties of the framework, valid for all conditions of loading, while the other takes into account the actual loading conditions in any given case. A numerical application to the case of a rectangular-panel truss is given.

1. Introduction

THE analysis of pin-jointed redundant plane frameworks by any of the classical methods involves the preparation of as many elastic equations as there are redundancies, and the simultaneous solution of these equations. In all but the simplest cases this means that a calculating machine is needed to cope with the number of significant figures required if important errors are to be avoided.

In the method described in the present paper there are no simultaneous equations to be solved; only simple arithmetical operations are used, and an accurate solution is obtainable by the use of a slide rule only. The method is particularly suitable for use when several loading conditions have to be investigated.

2. Notation

The following notation will be used in this paper:

A denotes cross-sectional area of a bar.

a denotes cross-sectional area of an additional bar forming part of an equivalent bar.

$$B = \frac{(L_{13})^3}{EA_{13}} + \frac{(L_{34})^3}{EA_{34}} + \frac{(L_{14})^2 L_{23}}{EA_{14}}.$$

$$B_1 = \frac{(L_{34})^3}{EA_{34}} + \frac{(L_{24})^3}{EA_{24}} + \frac{(L_{23})^2 L_{14}}{EA_{23}}.$$

C denotes the elastic constant for a panel or group of panels.

D denotes the relative axial displacement of cut ends in bar 23 produced by a unit tension applied in the line of the cut bar 23.

D_1 denotes the relative axial displacement of cut ends in bar 14 produced by a unit tension applied in the line of the cut bar 14.

E denotes Young's modulus.

F denotes the force in a bar.

f denotes the force in an additional bar.

$K = B/C$.

$K_1 = B_1/C$.

$K' = 1-K$.

$K_1' = 1-K_1$.

L denotes the length of bar; force in a bar.

L' denotes the extensibility (L/AE) of a bar.

P denotes the external force applied to a framework.

S' denotes the extensibility of an equivalent bar.

Δ denotes the relative axial displacement of two ends of a bar due to force F in the bar.

$\omega = L'_{14}/D_1 L_{23}$ or L'_{23}/DL_{14}.

e as suffix, refers to equivalent bar.

1, 2, as suffixes, refer to the two additional bars forming part of an equivalent bar.

34, 23, etc., as suffixes, refer to bars 34, 23, etc.

3. The single redundant panel

In the general case of any redundant panel not subject to external loading, such as 1234 (Fig. 1), the internal forces in the bars produced by

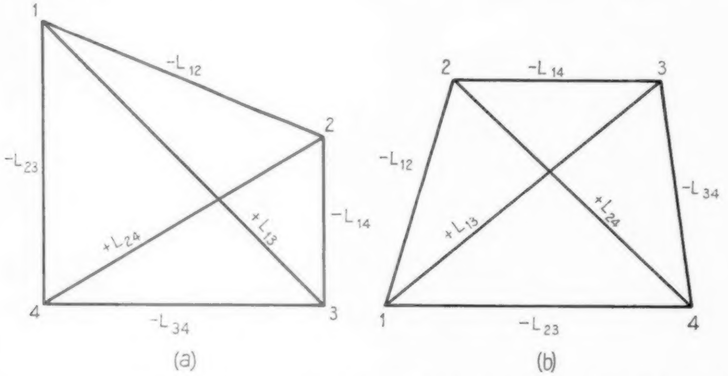


FIG. 1. Relative internal forces in bars of a redundant panel not subject to external loading.

a force in any one member are related to the lengths of the bars as indicated in Table 1 (a), in which the positive sign is used for tension. This

can be shown to be so by considering bar 24 to be removed and replaced by a force $+L_{24}$; this will create forces in the other bars having the values shown in the table. It will be noted that the force in bar 23 is $-L_{14}$, and that in bar 14 is $-L_{23}$; otherwise forces are proportional to the lengths of the bars. (These are the bars which, in a multi-panel structure, are common to adjacent panels, i.e. the vertical bars in the case of trusses or the horizontal bars in the case of towers.) In the special case of a rectangular panel, all forces are proportional directly to the lengths of the bars.

TABLE 1

Bar	12	34	13	24	23	14
(*a*)	$-L_{12}$	$-L_{34}$	$+L_{13}$	$+L_{24}$	$-L_{14}$	$-L_{23}$
(*b*)	$-KL_{12}$	$+K'L_{34}$	$-K'L_{13}$	$+KL_{24}$	$-KL_{14}$	$+L_{14}-KL_{23}$
(*c*)	$-K_1\,L_{12}$	$+K_1'L_{34}$	$+K_1\,L_{13}$	$-K_1'L_{24}$	$+L_{23}-K_1L_{14}$	$-K_1\,L_{23}$
(*d*)	$-\omega PL_{12}$	$-\omega PL_{34}$	$+\omega PL_{13}$	$+\omega PL_{24}$	$-\omega PL_{14}$	$(-\omega PL_{23})$

(*a*) Relative internal forces in bars of panel (no external loading).
(*b*) Internal forces in bars of panel (external loading, case I).
(*c*) Internal forces in bars of panel (external loading, case II).
(*d*) Induced forces in bars of panel.

If the bar 14 be cut, and equal and opposite forces $+L_{23}$ be applied in the direction of the cut bar, it may be shown that the relative axial displacement of the cut ends is

$$\frac{C}{L_{23}}, \tag{1}$$

in which
$$C = \sum \frac{L^3}{EA} + (L_{23})^2 \frac{L_{14}}{EA_{14}} + (L_{14})^2 \frac{L_{23}}{EA_{23}},$$

the summation referring to chord members and diagonals. The quantity C will be called the *elastic constant* for the panel. On the other hand, if a unit force is applied, instead of a force L_{23}, the relative axial displacement of the cut ends is

$$D_1 = \frac{C}{(L_{23})^2}. \tag{2}$$

The effect of external loading will now be considered. First, if the panel 1234, supported as shown in Fig. 2, is acted upon by a horizontal force P of amount L_{34} at the joint 1, the other external forces required for equilibrium are as shown in the figure. It may then be shown that if

$$B = \frac{(L_{13})^3}{EA_{13}} + \frac{(L_{34})^3}{EA_{34}} + \frac{(L_{14})^2 L_{23}}{EA_{14}},$$

$$K = \frac{B}{C}, \quad \text{and} \quad K' = 1-K,$$

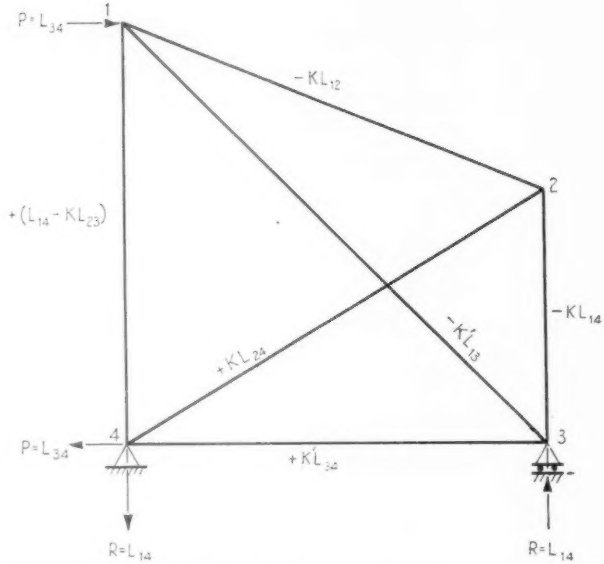


FIG. 2. Relative internal forces in bars of a redundant panel due to external loading on the left (case I).

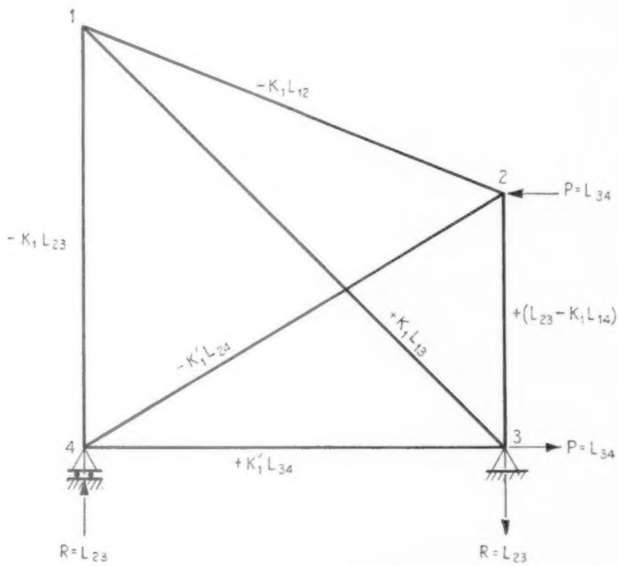


FIG. 3. Relative internal forces in bars of a redundant panel due to external loading on the right (case II).

the internal forces in the bars due to the external loading are as given in Table 1 (*b*). In the case of rectangular panels, a simplification is possible since $L_{14} = L_{23}$ so that the force in bar 14 may be written $K'L_{23}$ instead of $(L_{14}-KL_{23})$.

Secondly, if the same panel 1234, supported as shown in Fig. 3, is acted

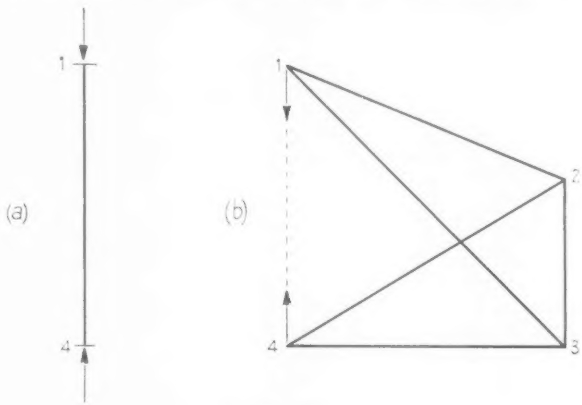


FIG. 4.

upon by a horizontal force P of amount L_{34} at joint 2, the internal forces in the bars are as given in Table 1 (*c*). Here

$$K_1 = \frac{B_1}{C}, \qquad K_1' = 1-K_1,$$

and

$$B_1 = \frac{(L_{34})^3}{EA_{34}}+\frac{(L_{24})^3}{EA_{24}}+\frac{(L_{23})^2 L_{14}}{EA_{23}}.$$

For rectangular panels, the force in bar 23 may be written $K_1' L_{14}$ instead of $(L_{23}-K_1 L_{14})$.

All bars in a panel may be replaced by one *equivalent bar* having the same elastic properties as the bars in the panel, considered as a group. Thus, the single panel illustrated in Fig. 1 *a* may be considered as composed of the two parts shown in Fig. 4. If equal and opposite unit forces are applied at the joints 1 and 4 (Fig. 4 *b*) in the direction of the missing bar the relative axial displacement of the joints is

$$\frac{C}{(L_{23})^2}-\frac{L_{14}}{EA_{14}} = D_1-L_{14}'. \tag{3}$$

It follows that to produce a unit relative axial displacement the force required is

$$\frac{1}{D_1-L_{14}'}. \tag{4}$$

To produce a unit relative axial displacement of the ends of the bar 14 itself (Fig. 4 *a*), the force required is

$$\frac{1}{L'_{14}}, \tag{5}$$

so that, for the complete panel (Fig. 1 *a*), the force needed to produce a unit relative axial displacement of the joints 1 and 4 is

$$\frac{1}{D_1 - L'_{14}} + \frac{1}{L'_{14}}, \quad \text{viz.} \quad \frac{D_1}{L'_{14}(D_1 - L'_{14})}. \tag{6}$$

Elastically, a single bar between joints 1 and 4 is equivalent to the six bars in the panel provided that

$$\frac{1}{S'_{14}} = \frac{D_1}{L'_{14}(D_1 - L'_{14})},$$

where S'_{14} is the L/EA value for the equivalent bar. Hence,

$$S'_{14} = L'_{14}\left(\frac{D_1 - L'_{14}}{D_1}\right). \tag{7}$$

If this equivalent bar is assumed to have the same length as the original bar 14, its cross-sectional area A_e must be given by

$$A_e = A_{14}\left(\frac{D_1}{D_1 - L'_{14}}\right). \tag{8}$$

Since the term in brackets is greater than unity, the equivalent bar may always be considered as made up of the original bar of area A_{14} together with another bar of the same length and of area a such that

$$a = A_{14}\left(\frac{L'_{14}}{D_1 - L'_{14}}\right), \tag{9}$$

the two bars acting in parallel between joints 1 and 4.

The effect of a force acting in the direction of such an equivalent bar may then be regarded as divided between the original bar and the additional bar in the ratio of their areas (Fig. 5). The force in the additional bar induces a set of internal forces in the other bars in the panel in accordance with Table 1 (*a*) and these will be called the *induced forces*. Thus, suppose a pull P to be applied to the panel at joints 1 and 4 in the direction of bar 14 (Fig. 6). Then the forces in the bars of the panel may be obtained as (*a*) a force in bar 14 of amount

$$F = \frac{P(D_1 - L'_{14})}{D_1}, \tag{10}$$

and (b) a set of forces in all other bars induced by a force in the direction 14 of amount

$$f = \frac{PL'_{14}}{D_1}. \tag{11}$$

The force $+PL'_{14}/D_1$ (tension) tends to separate the joints and this corresponds to the case in which bar 14 is cut and has its cut sections forced

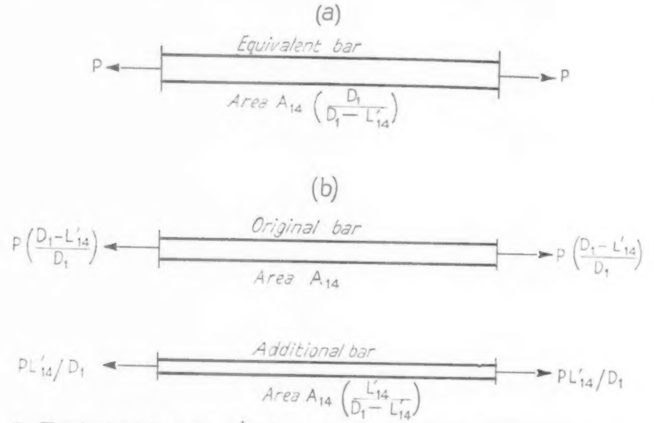


FIG. 5. Equivalent bar (a) considered as original bar plus additional bar (b).

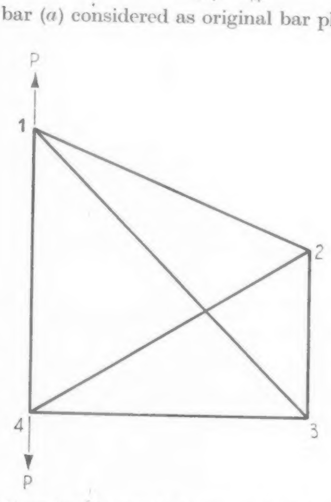


FIG. 6. Pull on bar 14 tends to separate joints 1 and 4.

apart, i.e. to the case of a compressive force $-PL'_{14}/D_1$ in the bar. By proportion from Table 1 (a) the forces induced in all the other bars may be written down as in Table 1 (d), in which $\omega = +L'_{14}/D_1 L_{23}$.

These forces are induced from left to right across the panel, but if P acts on joints 2 and 3 in the direction of the bar 23, the forces are induced from right to left. The values in Table 1 (*d*) still apply to this case provided that $\omega = +L'_{23}/DL_{14}$, where $D = C/L^2_{14}$. In this table the force quoted in brackets is the force in the additional bar with the sign reversed for the reason given.

4. Multi-panel trusses

In the case of a truss composed of a series of redundant panels, the end panel on the left may be reduced in the manner described above to an equivalent bar for which the S'_{23} value may be readily determined. If this value is then used in place of the L'_{14} value in the second panel, a second S'_{23} value may be found which will be the equivalent bar replacing the first two panels. In this way an equivalent bar (of value S'_{23}) may be found which can be used in replacement of any number of panels to the left. Similarly, another equivalent bar (of value S'_{14}) may be found which can be used in replacement of any number of panels to the right. The elastic constant for any such group of panels may be found as for a single panel. In doing this, a panel containing an equivalent bar is considered and the S' value for the equivalent bar is used in place of the L' value for the actual member there.

Any multi-panel truss may therefore, in general, be replaced by the chord members and diagonals of any one panel, together with two equivalent bars (of values S'_{23} and S'_{14}) replacing all panels to the left and right respectively. Such a system is called an *equivalent panel*. It follows that as many equivalent panels may be considered in any particular case as there are panels in the truss. The elastic constants of such equivalent panels are found in a manner similar to that which was described above for the single panel.

The forces acting at the joints of any equivalent panel are as follows: (*a*) a vertical force at joint 3/joint 4 equal to the algebraic sum of all the vertical forces to the right/left of the equivalent panel, and (*b*) two equal and opposite horizontal forces at joints 2 and 3/joints 1 and 4 exercising the same moment as that of all the forces to the right/left of the equivalent panel. These forces are provided automatically by the interaction between the panel considered and others on either side of it; such forces will be called the *conjugate forces* for the panel. They correspond to the external loadings considered in the case of the single panel. Hence, values of B, B_1, K, K_1, K', and K'_1 may be found for any given equivalent panel.

The conjugate forces acting on an equivalent panel produce forces in the bars in accordance with Tables 1 (*b*) and 1 (*c*), and these are called the

primary forces. They differ from the actual total forces by amounts which represent forces induced from other panels.

In any equivalent bar in an equivalent panel, a part of the primary force induces a set of internal forces in the bars of the next panel. Thus, if P is the primary force in equivalent bar 23 of the first panel on the left, the induced force in bar 14 of the second panel is, from (11), $P(-L'_{14}/D_1)$, where L'_{14} relates to bar 14 in the second panel and D_1 relates to the group of panels to the right (including the second panel). The induced forces in

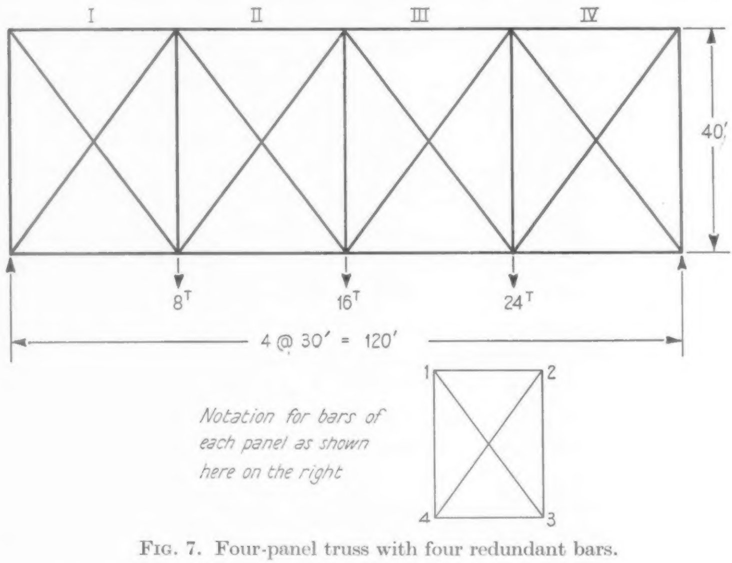


FIG. 7. Four-panel truss with four redundant bars.

the other bars of the panel are related as in Table 1 (d). Usually, the value ω referred to in this table is very small, which means that the induced forces are relatively very small and therefore effectively vanish after being induced from any one panel to the next. In this way forces are induced from left to right from panel to panel.

Similarly, if P is the primary force in the equivalent bar 14 of the last panel on the right, the induced force in bar 23 of the last-but-one panel is $P(-L'_{23}/D)$, where L'_{23} relates to the last-but-one panel and D relates to the group of panels to the left (including the last-but-one panel). The induced forces in the other bars are related as in Table 1 (d). Thus, forces are induced from right to left from panel to panel. In the case of a symmetrical frame loaded symmetrically, these two sets of induced forces are similar and do not require to be separately evaluated.

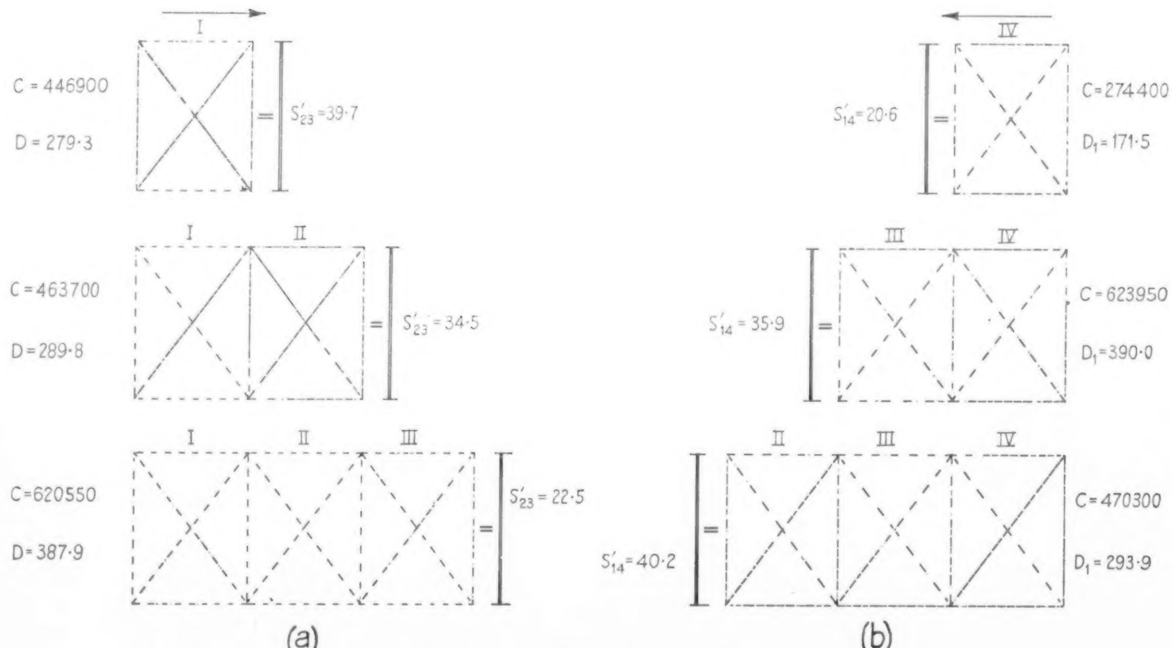
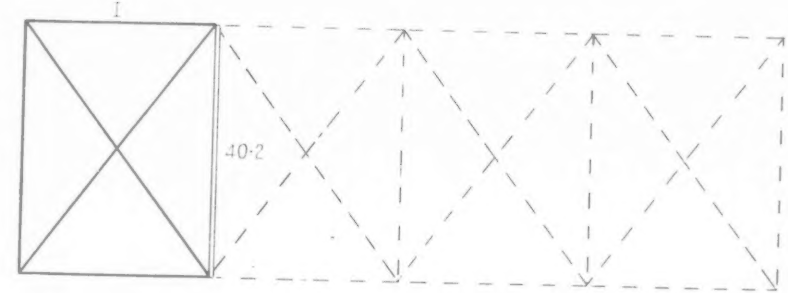


FIG. 8. Values of S' for equivalent bars replacing successive groups of panels (step 1).

For conjugate forces on left

For conjugate forces on right

I



$C = 434\,400$

$B_1 = 225400$

$K_1 = 0{\cdot}519$

$K_1' = 0{\cdot}481$

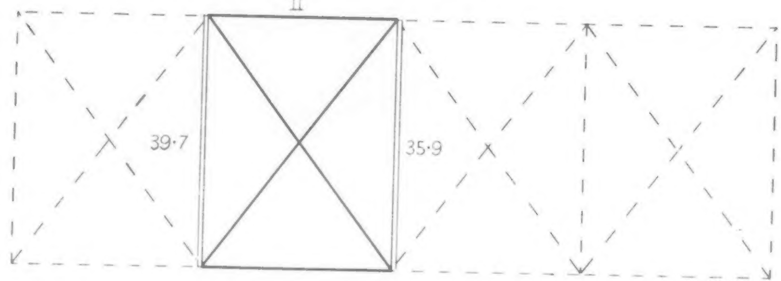
II

$C = 457020$

$B = 231520$

$K = 0{\cdot}507$

$K' = 0{\cdot}493$



$C = 457\,020$

$B_1 = 225\,500$

$K_1 = 0{\cdot}493$

$K_1' = 0{\cdot}507$

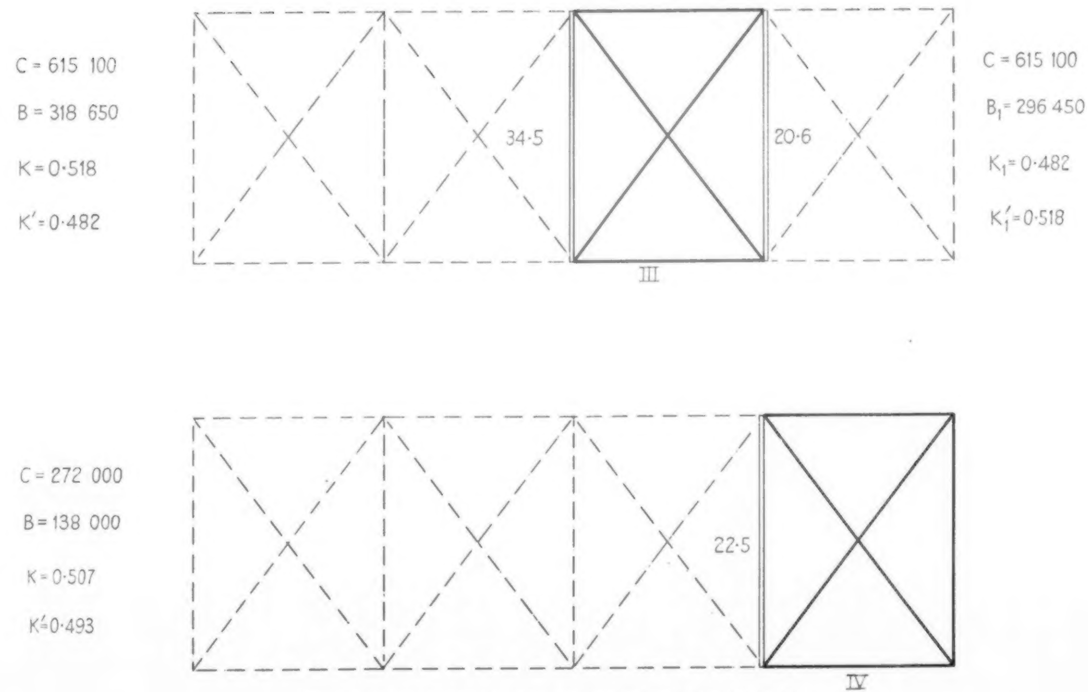


FIG. 9. Constants for the equivalent panels (step 2).

The *total force* in any bar is the sum of the primary force and forces induced from other panels.

5. Method of analysis

This is divided into two distinct stages which will be called the *structural* and the *stress* analyses, respectively.

The structural analysis involves the following two steps:

1. Evaluation of the equivalent bars for successive groupings of panels, working from left to right and also from right to left if the truss is unsymmetrical. In the case of a symmetrical truss the two sets of values will be similar.
2. Evaluation of the constants C, B, B_1, K, K', K_1, and K'_1 for each of the equivalent panels.

The structural analysis, which is an expression of the elastic properties of the truss, is valid for all conditions of loading.

In the stress analysis the forces in the bars due to any particular loading condition are determined according to the following steps:

3. Evaluation of the conjugate forces acting on each panel.
4. Determination of the primary forces in all bars from Tables 1 (*b*) and 1 (*c*).
5. Determination of all the induced forces working from left to right and also from right to left.
6. Summation of the primary forces and the induced forces to obtain the total forces in the bars.

6. Example

To illustrate the method of analysis using equivalent elastic systems, the case of a four-panel truss having four redundant bars (Fig. 7) will be considered. This example has been analysed elsewhere (**1**) by the two classical methods of virtual work and least work, and also by the Williot strain method, and the solutions reduced in each case as far as the four elastic equations. Particulars of the bars of this truss are given in Table 2.

TABLE 2

Values of L and L/A for all members of the truss

	Length (L ft.)	Length/Area (L/A in.$^{-1}$)			
Bar	All panels	Panel I	Panel II	Panel III	Panel IV
12	30	40	20	15	30
34	30	40	20	15	30
13	50	50	60	100	30
24	50	50	60	100	30
23	40	48	40	24	20
14	40	30	48	40	24

TABLE 3

Elastic constants for panel groups

	All panels	*Panel I*		*Panels I and II*		*Panels I–III*	
Bar	*L*	*L′*	*L′(L)²*	*L′*	*L′(L)²*	*L′*	*L′(L)²*
12	30	40	36 000	20	18 000	15	13 500
34	30	40	36 000	20	18 000	15	13 500
13	50	50	125 000	60	150 000	100	250 000
24	50	50	125 000	60	150 000	100	250 000
23	40	48	76 900	40	64 000	24	38 400
14	40	30	48 000	39·7	63 700	34·5	55 150
		Totals	446 900		463 700		620 550

Structural analysis. Step 1. The results of the first step (determination of the values of S' for the equivalent bars) are summarized in Fig. 8. For panel I alone, the values of $L'(L)^2$ are readily obtained (Table 3) from

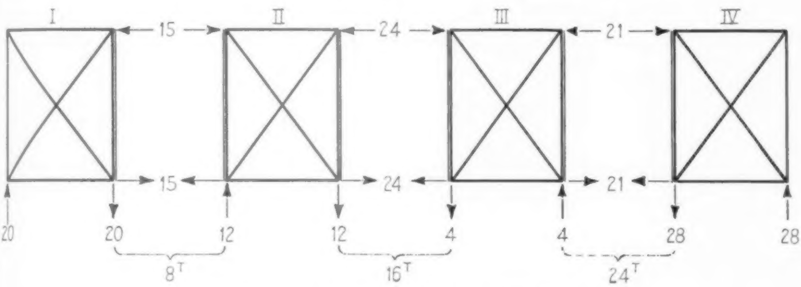


FIG. 10. Conjugate forces (step 3).

the given information. The sum, viz. 446,900, is the elastic constant C for the panel. Hence, from (2)

$$D = \frac{C}{(L_{14})^2} = \frac{446900}{(40)^2} = 279{\cdot}3,$$

and from (7)

$$S'_{23} = L'_{23}\left(\frac{D-L'_{23}}{D}\right) = \frac{48(279{\cdot}3-48)}{279{\cdot}3} = 48(0{\cdot}828) = 39{\cdot}7.$$

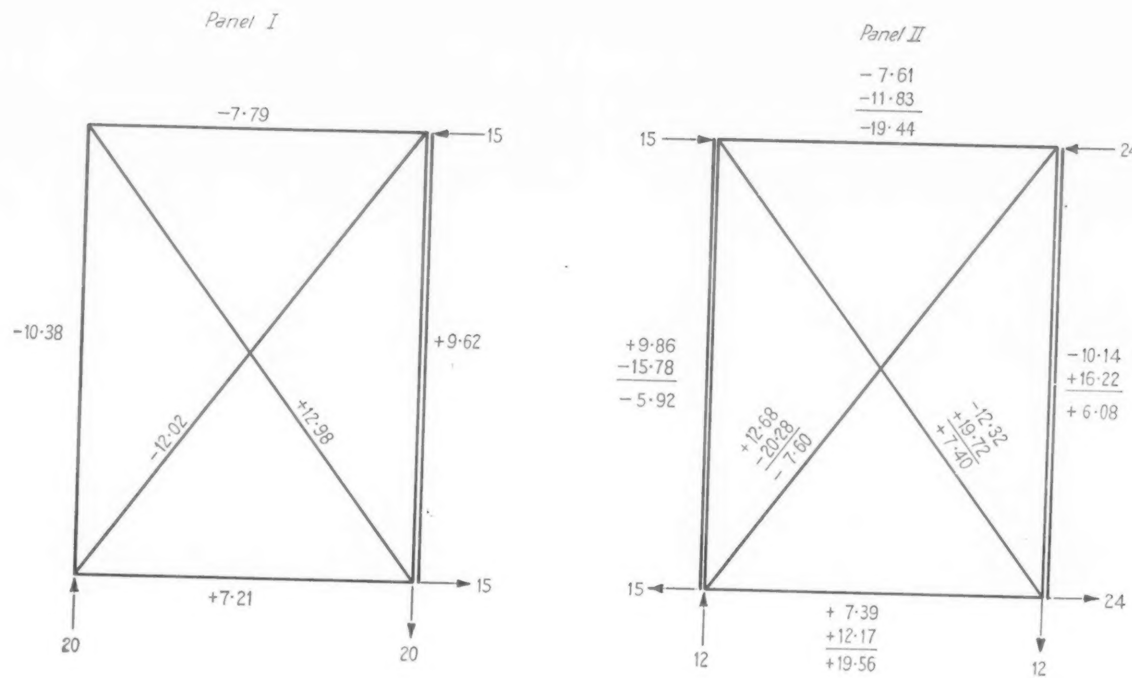
If this value is now used as the L'_{14} value in panel II (the other values being as already given) a similar process will give for panels I and II combined the following values:

$$C = 463{,}700; \quad D = 290; \quad S'_{23} = 40(0{\cdot}862) = 34{\cdot}5.$$

Similarly, the following are the values for panels I, II, and III combined:

$$C = 620{,}550; \quad D = 388; \quad S'_{23} = 24(0{\cdot}938) = 22{\cdot}5.$$

These values (of S'_{23}) are the values quoted in Fig. 8*a*; those given in


Panel I
−7·79
15
−10·38
+9·62
−12·02
+12·98
15
+7·21
20
20
Panel II
− 7·61
−11·83
−19·44
15
24
+9·86
−15·78
− 5·92
−10·14
+16·22
+ 6·08
+12·68
−20·28
− 7·60
−12·32
+19·72
+ 7·40
15
24
+ 7·39
+12·17
+19·56
12
12

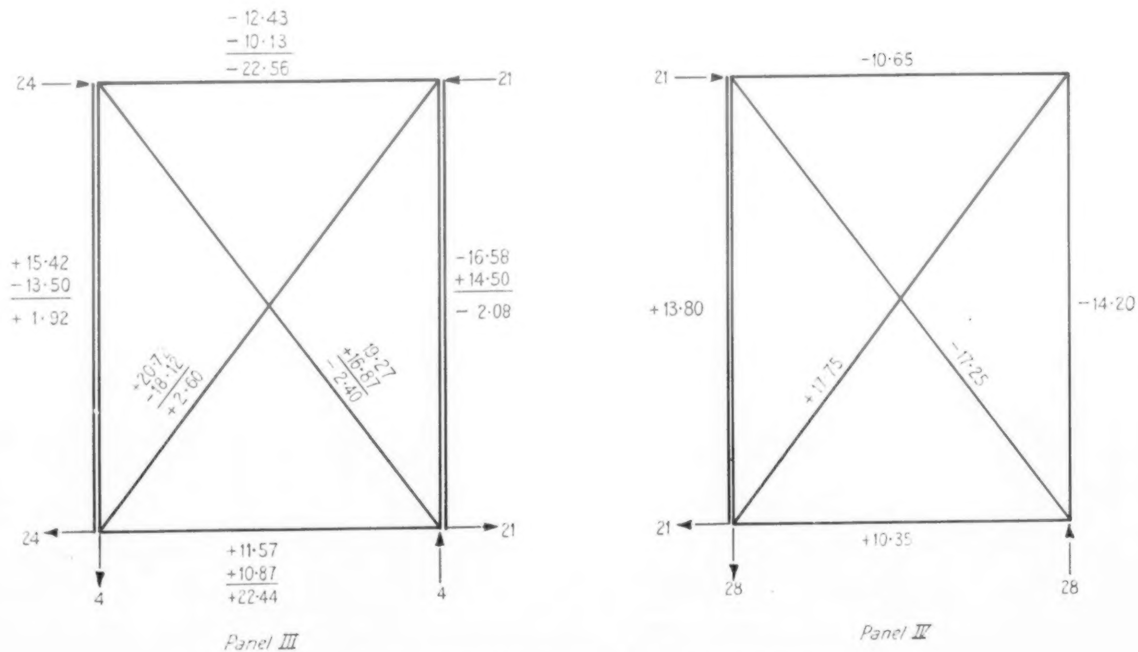


Fig. 11. Primary forces (step 4). (Plus tension, minus compression.)

Fig. 8 *b* are obtained from the opposite end of the truss and this time they are the S'_{14} values.

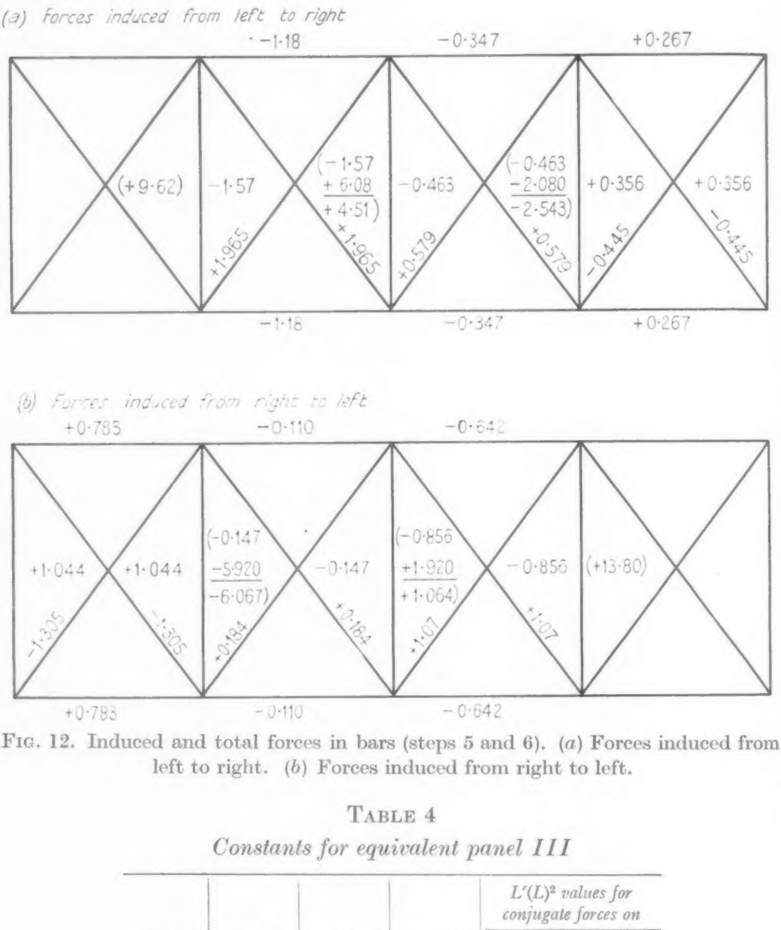


Fig. 12. Induced and total forces in bars (steps 5 and 6). (*a*) Forces induced from left to right. (*b*) Forces induced from right to left.

TABLE 4

Constants for equivalent panel III

Bar	L	L'	$L'(L)^2$	$L'(L)^2$ values for conjugate forces on: Left	Right
12	30	15	13 500	—	—
34	30	15	13 500	13 500	13 500
13	50	100	250 000	250 000	—
24	50	100	250 000	—	250 000
23	40	20·6	32 950	—	32 950
14	40	34·5	55 150	55 150	—
		Totals	615 100 (= C)	318 650 (= B)	296 450 (= B_1)

Step 2. The constants for the four equivalent panels are shown in Fig. 9. They may be determined conveniently from a tabulation such as Table 4

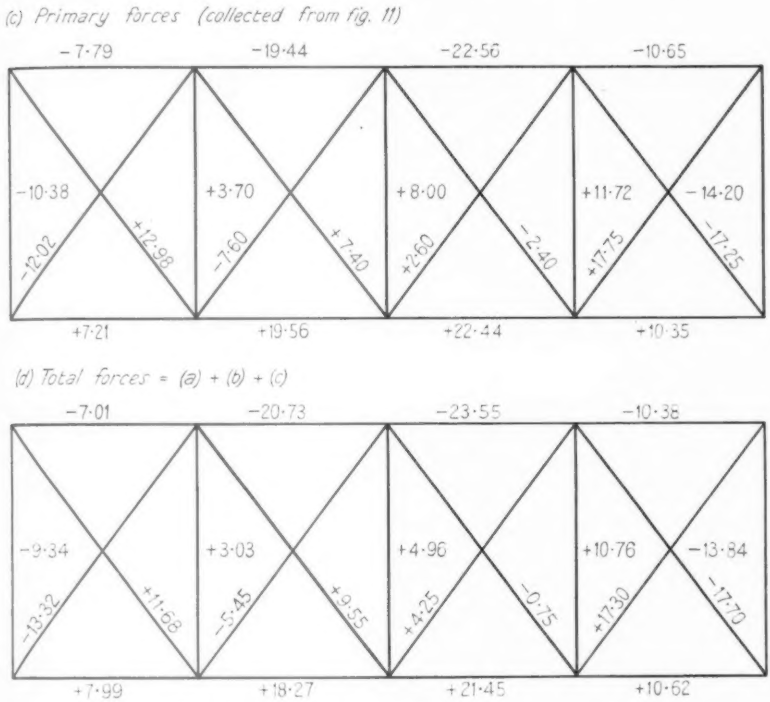


FIG. 12 (*contd.*). Induced and total forces in bars (steps 5 and 6). (*c*) Primary forces (collected from Fig. 11). (*d*) Total forces $= (a)+(b)+(c)$.

which deals with equivalent panel III and which is self-explanatory. It should be noted that both in Figs. 8 and 9 the dotted lines indicate members replaced by equivalent bars.

Stress analysis. Step 3. The conjugate forces are obtained by statics for the given loading and are shown in Fig. 10.

Step 4. The primary forces in the members of each panel may now be written down on a sketch as in Fig. 11, using Tables 1 (*b*) and 1 (*c*), and finally collected together as in Fig. 12*c*. In Fig. 11, for panels II and III, two sets of forces are shown. These are due to the conjugate forces on the right and left respectively; they have to be added to give the total primary forces in the bars.

Step 5. The forces induced from left to right are as shown in Fig. 12*a*.

The primary force $+9{\cdot}62$ ($= P$) in bar 23 of panel I induces a force in bar 14 of panel II of amount $P(-L'_{14}/D_1) = +9{\cdot}62(-48/293{\cdot}9) = -1{\cdot}57$. This gives rise to a set of internal forces in panel II, the value of the force in bar 23 being also $-1{\cdot}57$. In addition there is a primary force in this bar of $+6{\cdot}08$ giving a net value of $+4{\cdot}51$ which induces $-0{\cdot}463$ in bar 23, panel III, and so on. The forces induced from right to left are given in Fig. 12 *b*.

Step 6. The total forces in all bars are obtained by summation of primary forces (Fig. 12 *c*) and induced forces (Figs. 12 *a* and 12 *b*). They are shown in Fig. 12 *d*. For chord members and diagonals this is a straightforward operation. For vertical members the primary forces are already included in Figs. 12 *a* and 12 *b*, so that, for example, the total force in the middle vertical is obtained as

$$+4{\cdot}51-0{\cdot}463+1{\cdot}064-0{\cdot}147 = +4{\cdot}964.$$

7. Conclusions

1. The introduction of the concept of equivalent elastic systems provides a ready means of solving pin-jointed redundant plane frameworks without the necessity of formulating and solving simultaneous equations. All computations can be performed by slide rule.

2. The division of the analysis into two parts, of which one (the structural analysis) is valid for all conditions of loading, is particularly valuable in cases where the effect of a number of different loading conditions has to be investigated.

REFERENCE

1. C. A. Ellis, 'Simplified analysis of indeterminate frames', *Engineering News-Record*, **112** (1934), 534.

THREE-DIMENSIONAL TURBULENCE AND EVAPORATION IN THE LOWER ATMOSPHERE, I

By D. R. DAVIES (*The University, Sheffield*)

[Received 6 September 1949]

SUMMARY

The investigation is a first approach to the problem of turbulent diffusion from a finite rectangular lake. A formal solution in terms of Mathieu functions is obtained for the problem of vapour distribution above and down-wind of the lake on the basis of an assumed law of lateral diffusivity. The air-stream is assumed to be fully turbulent and directed parallel to the longitudinal length of the lake. The application of the formal solution to special cases awaits the complete tabulation of Mathieu functions.

1. Introduction

A THEORY to account for the rate of evaporation from plane saturated liquid surfaces into a fully turbulent air-stream was formulated by O. G. Sutton (**1**) and verified experimentally by Pasquill (**2**). The mathematical model of diffusion worked out by Sutton allows for the transfer of mass *vertically* only and, although quite satisfactory for assessing rates of evaporation, it does not give as good an agreement with experiment for the *distribution of vapour* above and down-wind from the evaporating surface. One improvement is suggested in the theoretical mechanism by the introduction of a *lateral* diffusivity. This is discussed by the author in a paper (**3**) in which an assumed simple power law of variation of lateral diffusivity with height is introduced to allow for this effect, and the resulting differential equations are solved to give the vapour distributions *over* plane liquid surfaces which have parabolic boundaries: it was necessary to assume in this paper that the parabolic area extended indefinitely down-wind, so that it was not possible to obtain the vapour distribution *down-wind* of a finite area.

The aim of this paper is to integrate these differential equations, subject to the conditions suitable for evaporation from a *rectangular* lake, of finite length and breadth, when the air-stream is directed parallel to the length of the lake.

As in the problem of the parabolic area, a solution is first obtained based on the assumption that wind speed and diffusivity are constant. In this first approximation suitable transformations are introduced which reduce the fundamental diffusion equation to that of heat flow in two dimensions. This is solved by a method similar to that used by McLachlan (**4**) for the problem of heat conduction in elliptical cylinders. The method is then

generalized to the problem of diffusion into an air-stream in which the wind speed and diffusivity, both vertical and lateral, obey simple power laws.

By an extension of another problem in heat conduction the vapour distribution *down-wind* of an evaporating area of any shape has been obtained. This completes the rectangle and also the previous parabola problem, on the basis of the hypothetical model of turbulence used by the present author.

2. The equation of diffusion and boundary conditions

If u denotes the mean wind velocity at height z, in the positive sense of the x-axis, and K_x, K_y, K_z are the coefficients of diffusivity in x, y, and z directions respectively, the equation satisfied by the concentration, i.e. mass per unit volume, $\chi(x,y,z)$ of vapour is

$$\frac{\partial\chi}{\partial t}+U\frac{\partial\chi}{\partial x}=\frac{\partial}{\partial x}\left[K_x\frac{\partial\chi}{\partial x}\right]+\frac{\partial}{\partial y}\left[K_y\frac{\partial\chi}{\partial y}\right]+\frac{\partial}{\partial z}\left[K_z\frac{\partial\chi}{\partial z}\right]. \tag{2.1}$$

In this equation K_x, K_y, and K_z are such that the rates per unit area at which χ is transferred across the planes x, y, z = constant, in the positive directions are $-K_x(\partial\chi/\partial x)$, $-K_y(\partial\chi/\partial y)$, and $-K_z(\partial\chi/\partial z)$.

Steady conditions are assumed and K_x is neglected, so that the diffusion equation (2.1) becomes

$$U\frac{\partial\chi}{\partial x}=\frac{\partial}{\partial y}\left[K_y\frac{\partial\chi}{\partial y}\right]+\frac{\partial}{\partial z}\left[K_z\frac{\partial\chi}{\partial z}\right]. \tag{2.2}$$

It is assumed that the vapour pressure of the evaporating liquid approaches its saturation value at the temperature of the surface as the surface is approached. This implies that the vapour concentration χ assumes a constant value χ_0 at the surface of the saturated area.

The appropriate boundary conditions then are

(i) $\chi/\chi_0 \to 1$, as $z\to 0$ for values of x and y inside the area,

(ii) $\lim_{z\to 0}\{K_z(\partial\chi/\partial z)\}\to 0$ for x and y outside the area,

(iii) $\chi\to 0$, $x>0$, $z\to\infty$,

(iv) $\chi\to 0$, $y\to\pm\infty$, $z\geqslant 0$,

(v) $\chi\to 0$, $x\to +0$, $z>0$.

3. Wind speed and diffusivities constant with height above evaporating surface

It is convenient to substitute non-dimensional variables defined by

$$x=\xi\alpha U,\qquad y=\eta\sqrt{(\alpha K_y)},\qquad z=\zeta\sqrt{(\alpha K_z)}, \tag{3.1}$$

where α is a scale factor of the dimensions of time, and U, K_y, K_z are constants. In these coordinates equation (2.2) becomes

$$\frac{\partial\chi}{\partial\xi} = \frac{\partial^2\chi}{\partial\eta^2}+\frac{\partial^2\chi}{\partial\zeta^2}, \tag{3.2}$$

with the boundary conditions

(i) $\chi/\chi_0 \to 1$, $l > \xi > 0$, $\eta^2 \leqslant c^2$, $\zeta \to 0$;

(ii) $\partial\chi/\partial\zeta \to 0$, for $\xi < l$, $\eta^2 \geqslant c^2$, $\zeta \to 0$ and for $\xi > l$, all η, $\zeta \to 0$;

(iii) $\chi \to 0$, $\xi \to +0$, $\zeta > 0$;

(iv) $\chi \to 0$, $\xi > 0$, $\zeta \to \infty$,

where $2c$ is the breadth of the rectangular lake in (ξ, η, ζ) coordinates and l the length of the lake.

Assume the existence of solutions of the type

$$\chi = e^{-\lambda\xi}g(\eta, \zeta) \tag{3.3}$$

and substitute in (3.2) which becomes

$$-\lambda\chi = \left[\frac{\partial^2}{\partial\eta^2}+\frac{\partial^2}{\partial\zeta^2}\right]\chi. \tag{3.4}$$

This equation, together with the form of the boundary conditions, suggests a transformation into elliptic coordinates (β, θ) defined by

$$\eta+i\zeta = c\cosh(\beta+i\theta). \tag{3.5}$$

Assume $$\chi = e^{-\lambda\xi}B(\beta)H(\theta), \tag{3.6}$$

and substitute in equation (3.2) which is then separable into

$$\frac{d^2B}{d\beta^2}-[A-\lambda c^2\cosh^2\beta]B = 0 \tag{3.7}$$

and $$\frac{d^2H}{d\theta^2}+[A-\lambda c^2\cos^2\theta]H = 0, \tag{3.8}$$

where A is the constant of separation.

The boundary conditions become

(i) $\chi/\chi_0 \to 1$, $l > \xi > 0$, $0 < \theta < \pi$, $\beta \to 0$;

(ii) $\partial\chi/\partial\theta = 0$, $l > \xi > 0$, $0 < \beta < \infty$, $\theta = 0$, π;

(iii) $\partial\chi/\partial\theta = 0$ at all points on ground level for $\xi > l$.

Periodic solutions of Mathieu's equation of 'elliptic' type (3.8) are given by

$$\mathrm{ce}_{2n}(\theta, \lambda) = \sum_{r=0}^{\infty} A_{2r}^{2n}\cos 2r\theta,$$

defined and extensively tabulated by E. L. Ince (**5**); the symmetry about $\theta = \frac{1}{2}\pi$ cuts out the ce_n of odd order. The solutions of the associated

'hyperbolic' type of Mathieu functions (3.7) have yet to be systematically tabulated. Definitions and references to existing tables are described by W. G. Bickley (**6**). The most suitable hyperbolic type of solution is given by W. G. Bickley in the above paper (equation 18.13) of the type

$$\frac{\mathrm{ce}_{2n}(0,\lambda)\mathrm{ce}_{2n}(\tfrac{1}{2}\pi,\lambda)}{A_0^2}\sum_r (-1)^r A_{2r}^{2n} J_r(ke^{-\beta})Y_r(ke^{\beta}),$$

where $k^2 = \lambda c^2/4$. Following the notation now in use these functions are denoted by $\mathrm{Fey}_{2n}(\beta,\lambda)$. They are known to converge uniformly in any finite region of the β-plane, including the origin; and they tend exponentially to zero for large β values. The existence of zeros of these functions when $\beta = 0$ follows from their asymptotic expansions at large λ values. Adopting the methods previously used by McLachlan (**7**) and Goldstein (**8**) in other physical problems, a formal solution expressed in terms of Mathieu functions is

$$\frac{\chi}{\chi_0} = 1 - \sum_{n=0}^{\infty} C_{2n}\,\mathrm{ce}_{2n}(\theta,\lambda)\mathrm{Fey}_{2n}(\beta,\lambda)e^{-\lambda\xi}, \tag{3.9}$$

where C_{2n} is a function of λ to be determined.

At the surface of the lake $\beta = 0$, and when $\xi > 0$, $\chi/\chi_0 \to 1$. Hence

$$\mathrm{Fey}_{2n}(0,\lambda) = 0. \tag{3.10}$$

Thus λ has those values $\lambda_{2n,m}$ which makes $\mathrm{Fey}_{2n}(0,\lambda)$ vanish, i.e. the parametric zeros of the function. The function (3.9) clearly satisfies the condition that $\partial\chi/\partial\theta = 0$ at $\theta = 0, \pi$. At the up-wind edge of the lake, the physical conditions may be represented mathematically by making $\chi/\chi_0 \to 0$ as $\xi \to +0$.

Hence,
$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} C_{2n}\,\mathrm{Fey}_{2n}(\beta,\lambda_{2n,m})\mathrm{ce}_{2n}(\theta,\lambda_{2n,m}) = 1. \tag{3.11}$$

This system of equations enables the C_{2n} to be evaluated. Use is made of the orthogonality theorem described by McLachlan (**7**, p. 176) and the development is similar to that used by him for the problem of heat conduction, except that integration with respect to β is taken from zero to infinity. A full explanation of this process is given in the general case described in section 4 of this paper. The formula obtained for C_{2n} is

$$C_{2n} = \frac{\int_0^{\infty} \mathrm{Fey}_{2n}(\beta,\lambda_{2n,m})[2A_0^{2n}\cosh 2\beta - A_2^{2n}]\,d\beta}{\int_0^{\infty} \mathrm{Fey}_{2n}^2(\beta,\lambda_{2n,m})[\cosh 2\beta - \Theta_{2n}]\,d\beta}, \tag{3.12}$$

where
$$\Theta_{2n} = A_0^{2n}A_2^{2n} + \sum_{r=0}^{\infty} A_{2r}^{2n}A_{2r+2}^{2n}. \tag{3.13}$$

The vapour distribution above a rectangular lake up to the second lateral edge is thus formally given by

$$\frac{\chi}{\chi_0} = 1 - \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} e^{-\lambda_{2n,m}\xi}\,\mathrm{Fey}_{2n}(\beta,\lambda_{2n,m})\mathrm{ce}_{2n}(\theta,\lambda_{2n,m})\times$$
$$\times\frac{\int\limits_0^{\infty}\mathrm{Fey}_{2n}(\beta,\lambda_{2n,m})[2A_0^{2n}\cosh 2\beta - A_2^{2n}]\,d\beta}{\int\limits_0^{\infty}\mathrm{Fey}_{2n}^2(\beta,\lambda_{2n,m})[\cosh 2\beta-\Theta_{2n}]\,d\beta}. \tag{3.14}$$

A general theorem in heat conduction theory may now be used to give the vapour distribution *down-wind* of the lake. If v is the temperature at time t at a point (x, y) the equation of heat conduction in two dimensions is

$$\frac{\partial v}{\partial t} = K\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right), \tag{3.15}$$

where K is the heat diffusivity of the substance.

By methods described by Carslaw (**9**) it may be shown that

$$v(x,y,t)$$
$$= \frac{1}{4\pi Kt}\int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty} f(x',y')[e^{-\{(x-x')^2+(y-y')^2\}/4Kt}+e^{-\{(x-x')^2+(y+y')^2\}/4Kt}]\,dx'dy' \tag{3.16}$$

satisfies the conditions

(i) $v = f(x,y)$ at time $t = 0$,

(ii) the solid is bounded by $y = 0$, this boundary being impervious to heat.

The corresponding solutions for vapour distribution are immediately obtained by putting $K = 1$ and $t = (\xi - l)$.

In order to calculate the distribution of vapour down-wind of the lake, the first step consists of evaluating numerically the distribution $\chi = f(\eta,\zeta)$ in the plane $\xi = l$, by using equation (3.14) above. Then for $\xi > l$,

$$\chi = \frac{1}{4\pi(\xi-l)}\int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty} f(\eta',\zeta')[e^{-\{(\eta-\eta')^2+(\zeta-\zeta')^2\}/4(\xi-l)}+e^{-\{(\eta-\eta')^2+(\zeta+\zeta')^2\}/4(\xi-l)}]\,d\eta'd\zeta'. \tag{3.17}$$

4. Wind speed and diffusivities variable with height above the surface: solutions up to the down-wind edge

Following a previous paper by the author (**3**), a practical model of the diffusion mechanism in a turbulent air-stream is obtained by writing

$$\frac{U}{U_1} = \left(\frac{z}{z_1}\right)^m, \tag{4.1}$$

$$K_z = a_z U_1^{1-n} z^{1-m}, \tag{4.2}$$

and

$$K_y = a_y U_1^{1-n} z^m. \tag{4.3}$$

In these expressions U_1 is the value of the mean wind speed at some standard height z_1; a_y, a_z, m, and n are constants for a given turbulent state and may be worked out numerically from simple meteorological measurements. They are described in detail in the above-mentioned paper and in the papers by O. G. Sutton. The relationship between m and n is $m = n/(2-n)$. A new development of the two-dimensional theory by K. L. Calder (**10**) yields different expressions for these constants and gives much improved agreement with observation, but the laws of variation of U and K_z with height in his theory are similar to equations (4.1) and (4.2) above. The present work may therefore be easily adjusted formally to incorporate the new constants.

The so-called conjugate power laws (4.1) and (4.2), although based on the assumption that the eddy shearing stresses are constant with height above ground level, are now generally accepted by meteorologists. As yet no other three-dimensional theory of atmospheric turbulence applicable to evaporation has been developed, and the form of equation (4.3) is chosen as it appears to be the only simple power law for K_y which, when inserted in the transport equation, leads to a tractable partial differential equation. However, the quite significant improvement in the calculated rates of evaporation and in vapour distribution at the down-wind edge of a saturated parabolic area, quoted by the present author (**3**), gives physical support for the use of equation (4.3). Further experimental evidence to show that the 'z^m law' may be used quite satisfactorily for evaporation problems is given in paper II of this series.

Equations (4.1), (4.2), and (4.3) are substituted in the transport equation (2.2) and the result simplified by writing

$$\xi = x, \qquad \eta = (U_1^n/a_y Z_1^m)^{\frac{1}{2}} y, \qquad \zeta = 2(U_1^n/a_z Z_1^m)^{\frac{1}{2}} z^{m+\frac{1}{2}}/(2m+1). \tag{4.4}$$

The fundamental equation becomes

$$\zeta^s \frac{\partial \chi}{\partial \xi} = \frac{\partial}{\partial \zeta}\left[\zeta^s \frac{\partial \chi}{\partial \zeta}\right] + \frac{\partial}{\partial \eta}\left[\zeta^s \frac{\partial \chi}{\partial \eta}\right], \tag{4.5}$$

where $s = (2-n)/(2+n)$.

Using elliptic coordinates $\eta + i\zeta = c\cosh(\beta + i\theta)$, equation (4.5) becomes

$$\frac{1}{c^2(\cosh^2\beta - \cos^2\theta)}\left[\frac{\partial^2}{\partial \beta^2} + \frac{\partial^2}{\partial \theta^2}\right]\chi +$$
$$+\frac{s}{c^2(\cosh^2\beta - \cos^2\theta)}\frac{1}{\sinh\beta\sin\theta}\left[\cosh\beta\sin\theta\frac{\partial}{\partial\beta} + \sinh\beta\cos\theta\frac{\partial}{\partial\theta}\right]\chi = \frac{\partial\chi}{\partial\xi}. \tag{4.6}$$

This equation is separable by writing

$$\chi = F(\xi)G(\beta)H(\theta).$$

Then it follows that $$F = e^{-\lambda\xi}, \tag{4.7}$$

$$\frac{d^2H}{d\theta^2}+s\cot\theta\frac{dH}{d\theta}+(A-\lambda c^2\cos^2\theta)H = 0, \tag{4.8}$$

and $$\frac{d^2G}{d\beta^2}+s\coth\beta\frac{dG}{d\beta}-(A-\lambda c^2\cosh^2\beta)G = 0. \tag{4.9}$$

Equation (4.8) has been discussed by E. L. Ince (**11**). It may be shown that it is possible to expand solutions of this equation as a series of ordinary Mathieu functions. Ince begins with an equation in the form

$$\frac{d^2u}{dz^2}+\left[a+2\theta\cos 2z-\frac{\vartheta(\vartheta-1)}{\sin^2 z}\right]u = 0, \tag{4.10}$$

which reduces to Mathieu functions when $\vartheta = 0$ or 1, and terms the periodic solutions as Associated Mathieu functions. He shows that (for $\vartheta > -\frac{1}{2}$) the solutions of (4.10) are also solutions of the integral equation,

$$u(z) = \mu\int\limits_0^\pi e^{k\cos z\cos s}\sin^\vartheta z\sin^\vartheta s\,u(s)\,ds, \tag{4.11}$$

the use of the symbol s in this equation being clearly distinguishable from its use in equations (4.8) and (4.9).

By using the orthogonality properties of the ordinary Mathieu functions it may be shown that

$$e^{k\cos z\cos s} = \sum \alpha_{2n}\,\mathrm{ce}_{2n}(z)\mathrm{ce}_{2n}(s). \tag{4.12}$$

The functions $\mathrm{ce}_{2r}(z)$ are known to satisfy the integral equation

$$\mathrm{ce}_{2r}(z) = \mu_{2r}\int\limits_0^\pi e^{k\cos z\cos s}\,\mathrm{ce}_{2r}(s)\,ds. \tag{4.13}$$

Substitute for $e^{k\cos z\cos s}$, then

$$\begin{aligned}\mathrm{ce}_{2r}(z) &= \mu_{2r}\sum_{n=0}^{\infty}\alpha_{2n}\,\mathrm{ce}_{2n}(z)\int\limits_0^\pi \mathrm{ce}_{2n}(s)\mathrm{ce}_{2r}(s)\,ds\\ &= \mu_{2r}\,\alpha_{2r}\,\pi\,\mathrm{ce}_{2r}(z),\end{aligned}$$

if the ce_r functions are normalized so that

$$\int\limits_0^\pi \mathrm{ce}_{2r}^2(z)\,dz = \pi.$$

Hence $$\alpha_{2r} = \frac{1}{\pi\mu_{2r}}. \tag{4.14}$$

Ince denotes the even periodic Associated Mathieu functions by $\mathrm{ce}_{2n}^{\vartheta}(z)$, where the index n has similar significance to that used in the ordinary Mathieu functions. Substitution of the expansion (4.12) in equation (4.11) gives

$$\mathrm{ce}_{2n}^{\vartheta}(z) = \frac{\mu_{2n}^{\vartheta}}{\pi}\int\limits_0^\pi\left[\sum_r\frac{1}{\mu_{2r}}\,\mathrm{ce}_{2r}(z)\mathrm{ce}_{2r}(s)\right]\sin^\vartheta z\sin^\vartheta s\,\mathrm{ce}_{2n}^{\vartheta}(s)\,ds.$$

The factors μ_{2n} and μ_{2n}^{ϑ} may be evaluated by methods similar to those given by McLachlan (**7**, p. 181). The equation for $\mathrm{ce}_{2n}^{\vartheta}(z)$ may be put into the form

$$\mathrm{ce}_{2n}^{\vartheta}(z) = \frac{\mu_{2n}^{\vartheta}}{\pi}\sin^{\vartheta} z \sum_r \frac{1}{\mu_{2r}}\,\mathrm{ce}_{2r}(z)\int_0^{\pi} \mathrm{ce}_{2n}^{\vartheta}(s)\mathrm{ce}_{2r}(s)\sin^{\vartheta}s\,ds$$

$$= \sin^{\vartheta}z \sum_r b_{2r}^{2n}\,\mathrm{ce}_{2r}(z), \tag{4.15}$$

where
$$b_{2r}^{2n} = \frac{\mu_{2n}^{\vartheta}}{\pi}\frac{1}{\mu_{2r}}\int_0^{\pi} \mathrm{ce}_{2n}^{\vartheta}(s)\sin^{\vartheta}s\,\mathrm{ce}_{2r}(s)\,ds.$$

This shows that it is possible to expand Associated Mathieu functions as a series of ordinary Mathieu functions, although it is not possible to obtain explicit expressions for the coefficients. In a numerical computation use must be made of numerical methods on the lines followed by Ince (**5**). For the purposes of the present development the expansion given above is convenient.

If $u = u_1(z)\sin^{\vartheta}z$ is substituted in equation (4.10), $u_1(z)$ is seen to satisfy

$$\frac{d^2u_1}{dz^2} + 2\vartheta\cot z\,\frac{du_1}{dz} + (a - \vartheta^2 + 2\theta\cos 2z)u_1 = 0. \tag{4.16}$$

This is identical with equation (4.8) if $2\vartheta = s$, $a - \vartheta^2 - 2\theta = A$, and $4\theta = -\lambda c^2$. Hence solutions of equation (4.8) may be written as a series of ordinary Mathieu functions

$$H_{2n}(\theta) = \sum_r b_{2r}^{2n}\,\mathrm{ce}_{2r}(\theta, \lambda). \tag{4.17}$$

Since equation (4.9) may be obtained from (4.8) by substituting $\theta = i\beta$, then similar expressions must exist for equation (4.9). Write solutions of (4.9) in the form

$$G_{2n}(\beta) = \sum_t c_{2t}^{2n}\,\mathrm{Fey}_{2t}(\beta, \lambda). \tag{4.18}$$

A formal solution to the problem is then

$$\frac{\chi}{\chi_0} = 1 - \sum_n C_{2n}(\lambda)H_{2n}(\theta, \lambda)G_{2n}(\beta, \lambda)e^{-\lambda\xi}, \tag{4.19}$$

where $C_{2n}(\lambda)$ is to be determined by the boundary conditions.

It is now necessary to consider an orthogonality theorem for the Associated Mathieu functions on similar lines to that described by McLachlan. Let $\phi_n(\theta, \lambda)$ be a solution of the Associated Mathieu equation

$$\frac{d^2\phi_n}{d\theta^2} + \left[a_{n,m} - \lambda_{n,m}c^2\cos 2\theta - \frac{\vartheta(\vartheta-1)}{\sin^2\theta}\right]\phi_n = 0, \tag{4.20}$$

and let $\lambda_{n,m}$ be such that $\psi_n(\beta, \lambda_{n,m}) = 0$ for $\beta = 0$, where $\psi_n(\beta, \lambda)$ is a solution of the corresponding equation

$$\frac{d^2\psi_n}{d\beta^2} - \left[a_{n,m} - \lambda_{n,m} c^2 \cosh 2\beta - \frac{\vartheta(\vartheta-1)}{\sinh^2\beta}\right]\psi_n = 0. \tag{4.21}$$

Write $\zeta_{n,m} = \phi_n \psi_n$, then for $(a_{n,m}, \lambda_{n,m})$

$$\frac{\partial^2\zeta_{n,m}}{\partial\beta^2} + \frac{\partial^2\zeta_{n,m}}{\partial\theta^2} + \left\{c^2\lambda_{n,m}(\cosh 2\beta - \cos 2\theta) - \vartheta(\vartheta-1)\left[\frac{1}{\sin^2\theta} - \frac{1}{\sinh^2\beta}\right]\right\}\zeta_{n,m} = 0. \tag{4.22}$$

Similarly it follows for $(a_{p,r}, \lambda_{p,r})$

$$\frac{\partial^2\zeta_{p,r}}{\partial\beta^2} + \frac{\partial^2\zeta_{p,r}}{\partial\theta^2} + \left\{c^2\lambda_{p,r}(\cosh 2\beta - \cos 2\theta) - \vartheta(\vartheta-1)\left[\frac{1}{\sin^2\theta} - \frac{1}{\sinh^2\beta}\right]\right\}\zeta_{p,r} = 0. \tag{4.23}$$

Now multiply (4.22) by $\zeta_{p,r}$, (4.23) by $\zeta_{n,m}$, and subtract,

$$\frac{\partial}{\partial\beta}\left[\zeta_{p,r}\frac{\partial\zeta_{n,m}}{\partial\beta} - \zeta_{n,m}\frac{\partial\zeta_{p,r}}{\partial\beta}\right] + \frac{\partial}{\partial\theta}\left[\zeta_{p,r}\frac{\partial\zeta_{n,m}}{\partial\theta} - \zeta_{n,m}\frac{\partial\zeta_{p,r}}{\partial\theta}\right] + $$
$$+ c^2(\cosh 2\beta - \cos 2\theta)(\lambda_{n,m} - \lambda_{p,r})\zeta_{n,m}\zeta_{p,r} = 0. \tag{4.24}$$

Integrate (4.24) with respect to β from 0 to ∞ and θ from 0 to 2π,

$$\int\limits_0^{2\pi}\left[\zeta_{p,r}\frac{\partial\zeta_{n,m}}{\partial\beta} - \zeta_{n,m}\frac{\partial\zeta_{p,r}}{\partial\beta}\right]_0^\infty d\theta + \int\limits_0^\infty\left[\zeta_{p,r}\frac{\partial\zeta_{n,m}}{\partial\theta} - \zeta_{n,m}\frac{\partial\zeta_{p,r}}{\partial\theta}\right]_0^{2\pi} d\beta +$$
$$+ 2c^2(\lambda_{n,m} - \lambda_{p,r})\int\limits_0^\infty\int\limits_0^{2\pi}(\cosh 2\beta - \cos 2\theta)\zeta_{n,m}\zeta_{p,r}\, d\beta d\theta = 0. \tag{4.25}$$

Take $$\phi_n = \mathrm{ce}_{2n}^{\vartheta}(\theta) = \sin^{\vartheta}\theta \sum_r b_{2r}^{2n}\,\mathrm{ce}_{2r}(\theta)$$

and $$\chi_n = \mathrm{Fey}_{2n}^{\vartheta}(\beta) = \sinh^{\vartheta}\beta \sum_t c_{2t}^{2n}\,\mathrm{Fey}_{2t}(\beta).$$

By using the asymptotic formulae for $\mathrm{Fey}_{2n}(\beta)$ given by McLachlan (**7**, p. 220) it may be shown that $\mathrm{Fey}_{2n}^{\vartheta}(\beta) \to 0$ as $\beta \to \infty$. By virtue of this fact, the periodicity in θ and the use of equation (4.28), the first two integrals vanish. Hence if $p \neq n$,

$$\int\limits_0^\infty\int\limits_0^{2\pi}(\cosh 2\beta - \cos 2\theta)\zeta_{n,m}\zeta_{p,r}\, d\beta d\theta = 0. \tag{4.26}$$

This holds also if $p = n$, $r \neq m$, but if $p = n$ and $r = m$,

$$\int\limits_0^\infty\int\limits_0^{2\pi}[\mathrm{Fey}_{2n,m}^{\vartheta}(\beta)]^2[\mathrm{ce}_{2n,m}^{\vartheta}(\theta)]^2(\cosh 2\beta - \cos 2\theta)\, d\beta d\theta \neq 0. \tag{4.27}$$

Now apply the boundary conditions to equation (4.19). At the lake surface $\beta = 0$ and when $\xi > 0$, $(\chi/\chi_0) = 1$. Hence

$$\sum_n \mathrm{Fey}_{2n}(0, \lambda) = 0, \tag{4.28}$$

and thus as in section 3, λ has those values $\lambda_{2n,m}$ which make each $\mathrm{Fey}_{2n}(0,\lambda)$ vanish.

It may be shown that near the surface K_z behaves like $\sin^{3s-1}\theta$ ($3s > 1$ in the atmosphere) and that $\partial\chi/\partial z$ is proportional to $\partial\chi/\partial\theta$. Therefore, for points at ground level outside the wet surface the function (4.19) satisfies the requirement that

$$\lim_{z\to 0}\left[K_z\frac{\partial\chi}{\partial z}\right] = 0 \quad \text{at} \quad \theta = 0, \pi.$$

At the up-wind lateral edge of the lake the physical conditions give $\chi/\chi_0 \to 0$ as $\xi \to +0$. Hence the system of equations to evaluate C_{2n} follow,

$$1 = \sum_n\sum_m C_{2n}\, H_{2n}(\theta)G_{2n}(\beta). \tag{4.29}$$

Employing the orthogonality theorem by multiplying each side of (4.29) by

$$H_{2p}(\theta,\lambda_{2p,r})G_{2p}(\beta,\lambda_{2p,r})(\cosh 2\beta-\cos 2\theta)\sinh^{2\vartheta}\beta\sin^{2\vartheta}\theta,$$

and integrating with respect to θ from 0 to 2π and with respect to β from 0 to infinity, then the right-hand side vanishes except when $p = n$, $r = m$. Hence,

$$\int_0^\infty\int_0^{2\pi} H_{2n}(\theta,\lambda_{2n,m})G_{2n}(\beta,\lambda_{2n,m})[\cosh 2\beta-\cos 2\theta]\sinh^{2\vartheta}\beta\sin^{2\vartheta}\theta\, d\beta d\theta$$

$$= C_{2n}\int_0^\infty\int_0^{2\pi}[\mathrm{Fey}^{\vartheta}_{2n}(\beta,\lambda_{2n,m})]^2[\mathrm{ce}^{\vartheta}_{2n}(\theta,\lambda_{2n,m})]^2(\cosh 2\beta-\cos 2\theta)\, d\beta d\theta. \tag{4.30}$$

The vapour distribution above a rectangular lake up to the second lateral edge is thus given in terms of Mathieu functions by

$$\frac{\chi}{\chi_0} = 1-\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} e^{-\lambda_{2n,m}\xi}\sum_r b^{2n}_{2r}\,\mathrm{ce}_{2r}(\theta,\lambda_{2n,m})\sum_t c^{2n}_{2t}\,\mathrm{Fey}_{2t}(\beta,\lambda_{2n,m})\times$$

$$\times\frac{\displaystyle\int_0^\infty\int_0^{2\pi}\sum_r b^{2n}_{2r}\,\mathrm{ce}_{2r}(\theta,\lambda_{2n,m})\sum_t c^{2n}_{2t}\,\mathrm{Fey}_{2t}(\beta,\lambda_{2n,m})\times(\cosh 2\beta-\cos 2\theta)\sinh^{\vartheta}\beta\sin^{\vartheta}\theta\, d\beta d\theta}{\displaystyle\int_0^\infty\int_0^{2\pi}\Big[\sum_r b^{2n}_{2r}\,\mathrm{ce}_{2r}(\theta,\lambda_{2n,m})\Big]^2\Big[\sum_t c^{2n}_{2t}\,\mathrm{Fey}_{2t}(\beta,\lambda_{2n,m})\Big]^2(\cosh 2\beta-\cos 2\theta)\, d\beta d\theta}. \tag{4.31}$$

5. Wind speed and diffusivities variable with height above the surface: solution down-wind of the lake

Using the notation of section 4, write

$$\zeta\xi^{-\frac{1}{2}} = z, \qquad \eta\xi^{-\frac{1}{2}} = y,$$

and

$$\chi = \xi^{n_0}\psi(y,z), \tag{5.1}$$

where the significance of the symbols y and z is clearly distinguishable from their use in earlier sections. Substituting these in (4.5) it becomes

$$n_0\psi-\tfrac{1}{2}\left(y\frac{\partial\psi}{\partial y}+z\frac{\partial\psi}{\partial z}\right)=\frac{\partial^2\psi}{\partial y^2}+\frac{1}{z^s}\frac{\partial}{\partial z}\left(z^s\frac{\partial\psi}{\partial z}\right). \tag{5.2}$$

Putting $\psi = Y(y)Z(z)$ in (5.2) and separating the variables, we have

$$\frac{Y''}{Y}+\frac{y}{2}\frac{Y'}{Y}-n_0=-\left\{\frac{(1/z^s)(d/dz)(z^sZ')+\frac{1}{2}zZ'}{Z}\right\}=-\alpha,$$

where accents denote derivatives. It follows that

$$Y''+\tfrac{1}{2}yY'=-\beta Y \tag{5.3}$$

and

$$\frac{1}{z^s}\frac{d}{dz}(z^sZ')+\tfrac{1}{2}zZ'=\alpha Z, \tag{5.4}$$

where $\alpha-\beta=n_0$.

A suitable special solution of equation (5.3), when $\beta=\frac{1}{2}$, is

$$Y=e^{-y^2/4}. \tag{5.5}$$

To fit the conditions of the problem, choose $n_0=-1$, then

$$\alpha=\beta+n_0=-\tfrac{1}{2}.$$

Rewriting equation (5.4) in the form

$$Z''+\frac{s}{z}Z'=-\tfrac{1}{2}(zZ'+Z), \tag{5.6}$$

and substituting a series solution in the form

$$Z=\sum a_n z^n,$$

the relationship between the coefficients of z in the general case is found to be

$$a_n=\frac{-(n-1)}{2n(n-1+s)}a_{n-2}. \tag{5.7}$$

The modulus of the ultimate ratio of convergence is $z^2/2n$, so the series is absolutely convergent for all values of z. In the present problem even-power series solutions are required, and in the following work these will be denoted by Z.

To obtain the vapour distribution down-wind of the lake the method described by Carslaw (**9**) may be followed. Let $\xi_1=\xi-l$; then suppose the distribution in the vertical plane at $\xi_1=0$ has been calculated from the solution in section 4 or is known from experiment. Suppose that at $\xi_1=0$, $\chi=f(\eta,\zeta)$. Using the special solutions of (5.3) and (5.4) found above, the vapour concentration at any point (η,ζ) for $\xi_1>0$ must be of the form

$$\chi=\frac{A}{\xi_1}\int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty}f(\eta',\zeta')\left[e^{-(\eta-\eta')^2/4\xi_1}Z\left(\frac{\zeta-\zeta'}{\sqrt{\xi_1}}\right)+e^{-(\eta-\eta')^2/4\xi_1}Z\left(\frac{\zeta+\zeta'}{\sqrt{\xi_1}}\right)\right]d\eta'd\zeta', \tag{5.8}$$

where A is a constant to be determined.

Consider the first term in the integrand. It is sufficient to show that this may be reduced to $\frac{1}{2}f(\eta,\zeta)$ for $\zeta > 0$ when $\xi_1 > 0$. Write

$$\eta-\eta' = -\surd\xi_1 y$$

and

$$\zeta-\zeta' = -\surd\xi_1 z.$$

The first term in (5.8) becomes

$$A\int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty} f(\eta+\surd\xi_1 y,\ \zeta+\surd\xi_1 z)e^{-y^2/4}Z(z)\,dydz.$$

Let $\xi_1 \to 0$, then this reduces to

$$Af(\eta,\zeta)\int\limits_{-\infty}^{\infty} e^{-y^2/4}\,dy\int\limits_{0}^{\infty} Z\,dz = Af(\eta,\zeta)2\pi^{\frac{1}{2}}F(s),$$

where $F(s) = \int\limits_{0}^{\infty} Z(z,s)\,dz$. Hence the final solution for $\xi > l$ is

$$\chi = \frac{1}{4\pi^{\frac{1}{2}}F(s)(\xi-l)}\int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty} f(\eta',\zeta')\left[e^{-(\eta-\eta')^2/4(\xi-l)}Z\left\{\frac{\zeta-\zeta'}{\surd(\xi-l)}\right\}+\right.$$
$$\left.+e^{-(\eta-\eta')^2/4(\xi-l)}Z\left\{\frac{\zeta+\zeta'}{\surd(\xi-l)}\right\}\right]d\eta' d\zeta'. \qquad (5.9)$$

6. Discussion

It must be borne in mind that the present paper describes a mathematical solution to the problem of turbulent transfer from a rectangular lake, on the basis of an *assumed* power law of variation of lateral diffusivity with height above ground. The application of this and other assumed laws to diffusion from a continuous point source and to evaporation problems is discussed in paper II of the series.

It has been shown experimentally by Pasquill (**2**) that Sutton's theory is adequate to account for the rate of transmission of heat from a flat surface by turbulent action. It therefore appears probable, if the temperature difference between the surface and the air-stream is not large, that the mathematical solution obtained in this paper may be applied to give the distribution of temperature above and down-wind of a finite heated rectangular plate kept at constant uniform temperature, surrounded by a plane surface impervious to heat. In this connexion χ in the evaporation problem should be replaced by $\rho C_p t$, where C_p is the specific heat of air at constant pressure, t is the difference in temperature between plate surface and free air-stream, ρ the air density.

The work of computing numerically the general solution giving the vapour distribution above a rectangular lake cannot be undertaken until

the $\mathrm{Fey}_{2n}(\beta, \lambda)$ functions have been systematically tabulated. The complex nature of equation (4.31) shows that even when these are available much labour will be involved in the evaluation, so for the time being the solution, like many other problems involving Mathieu functions, has a formal value only.

However, the analysis of section 5 is applicable to any evaporating area and gives a method of completing the problem of vapour distribution from a parabolic area (**3**). Here the solution involves functions which may be readily computed; and the vapour distribution, in a vertical plane perpendicular to the axis of the parabola at the down-wind edge worked out numerically. The formulae of section 5 now apply and the full effects of the *assumed* law of variation of lateral diffusion on the calculated distribution *down-wind* of the area may be estimated. It is proposed to carry out this numerical investigation as time permits.

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THREE-DIMENSIONAL TURBULENCE AND EVAPORATION IN THE LOWER ATMOSPHERE, II

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SUMMARY

This paper gives a determination of a power law for the variation of the coefficient of lateral diffusivity with height above ground, and contains a discussion of the application of a previously used law to evaporation problems. The index in the power law is determined by comparison of theory with well-known results giving the variation of peak concentration (i.e. weight of suspended matter per unit volume) with distance down-wind from a continuous point source, such as a single static smoke generator, producing a cloud of air-borne particles of matter. The applications of this empirical law of diffusivity, and of one previously used, to evaporation problems are compared and discussed.

1. Introduction

THE problems of evaporation and diffusion of vapour, into the earth's turbulent air-stream, above and down-wind from saturated liquid surfaces of parabolic and rectangular shape have been solved by the author. A solution of the former problem was published by him in 1947 (**1**) and a solution of the latter appears as paper I of this series (**2**). The two-dimensional model of turbulence used by O. G. Sutton (**3**), and recently perfected by K. L. Calder (**4**) was adopted and a *three-dimensional* mechanism set up by the introduction of an *assumed* law of lateral eddy diffusivity (see **1**, equation 4.4). No other three-dimensional theory, applicable to evaporation problems, exists, and the choice of power law for the lateral effect was governed by mathematical convenience and physical plausibility; it was supported by the very significant improvement (compared with two-dimensional theory) in the agreement between theory and experiment for rates of evaporation and diffusion of vapour (**1**).

For the purposes, however, of obtaining a solution to the problem of diffusion of air-borne particles of matter from a continuous point source, such as a single static smoke generator, knowledge of a more exact power law is necessary. At the present time it appears that the only method of searching for this law must involve the following procedure: (i) one must accept the experimentally established laws of variation for wind speed and vertical diffusivity from two-dimensional theory; (ii) one must assume possible laws of variation of lateral diffusivity with height; (iii) one must

obtain the corresponding continuous point source solutions and (iv) compare with known results of experiment.

This method has been followed in this paper and an appropriate law of variation has been determined.

2. The equation of diffusion, incorporating the lateral effect

If U denotes the mean wind speed at a vertical height z above ground level in the positive sense of the x-axis of rectangular axes Ox, Oy, Oz, and K_y, K_z are the usual coefficients of eddy diffusivity in the y and z directions, the equation satisfied by the concentration $\chi(x,y,z)$, i.e. the weight of liquid vapour per unit volume, is

$$U\frac{\partial \chi}{\partial x} = \frac{\partial}{\partial y}\left[K_y\frac{\partial \chi}{\partial y}\right]+\frac{\partial}{\partial z}\left[K_z\frac{\partial \chi}{\partial z}\right], \tag{2.1}$$

when steady conditions are assumed and diffusion in the x direction is neglected.

At this stage use is made of the so-called 'conjugate power law' which appears to be generally accepted by meteorologists, namely

$$\frac{U}{U_1} = \left(\frac{z}{z_1}\right)^m, \tag{2.2}$$

and

$$K_z = a_z U_1^{1-n} z^{1-m}. \tag{2.3}$$

In these expressions U_1 is the value of the mean wind speed at some standard height z_1; a_z, m, and n are constants for a given turbulent state and may be evaluated numerically from meteorological measurements. They are described in detail by O. G. Sutton (**3**). The relationship between m and n is $m = n/(2-n)$. The recent development of the two-dimensional theory by K. L. Calder (**4**) yields new expressions for these constants and gives much improved agreement with experiment, but the laws of variation of U and K_z with height in his theory are similar to equations (2.2) and (2.3) above. The present work, therefore, incorporates the new constants.

As equations (2.2) and (2.3) represent a mathematical model of diffusion which agrees well with experiment for two-dimensional problems, it appears that a three-dimensional model will best be defined by accepting (2.2) and (2.3) and writing

$$K_y = a_y U_1^{1-n} z^{\alpha}, \tag{2.4}$$

where α is an index which has to be determined.

The coefficient a_y will depend on the index α, as K_y has to be kept dimensionally correct. The expression for a_y may be defined by analogy with a_z as previously suggested by the author (**1**, equation (4.5)), and support for this procedure comes from the fact that in the case of evaporation from parabolic areas it yields results in accord with experiment.

Substitution of equations (2.2)–(2.4) into (2.1) leads to an equation of the form

$$z^m \frac{\partial \chi}{\partial x} = A \frac{\partial}{\partial z}\left(z^{1-m}\frac{\partial \chi}{\partial z}\right) + B \frac{\partial}{\partial y}\left(z^{\alpha}\frac{\partial \chi}{\partial y}\right), \tag{2.5}$$

where $A = a_z z_1^m/U_1^n$ and $B = a_y z_1^m/U_1^n$.

3. Evaluation of an empirical power law of variation for K_y with height above ground, from well-known results for a continuous point source

Write

$$\frac{y}{x^{1/q}} = u, \qquad \frac{z^\beta}{x^{1/q}} = v, \tag{3.1}$$

and

$$\chi = x^{n_0}\psi(u, v), \tag{3.2}$$

where the constants β, q, and n_0 are to be determined, and the equation (2.5) becomes

$$n_0 x^{n_0-1}\psi - \frac{x^{n_0-1}}{q}\left(u\frac{\partial \psi}{\partial u} + v\frac{\partial \psi}{\partial v}\right) = B(vx^{1/q})^{(\alpha-m)/\beta}x^{(n_0-2/q)}\frac{\partial^2 \psi}{\partial u^2} +$$

$$+A\left[\beta^2(vx^{1/q})^{(2\beta-2m-1)/\beta}x^{(n_0-2/q)}\frac{\partial^2 \psi}{\partial v^2} + \right.$$

$$\left. +\beta(\beta-m)(vx^{1/q})^{(\beta-2m-1)/\beta}x^{(n_0-1/q)}\frac{\partial \psi}{\partial v}\right]. \tag{3.3}$$

The partial differential equation (2.5) involving three variables x, y, and z may now be reduced to one containing two only, u and v, by arranging the indices of x on both sides of equation (3.3) to be equal to (n_0-1) and thus cancelling out x, i.e.

$$\frac{\alpha-m}{\beta q} + n_0 - \frac{2}{q} = n_0 - 1, \tag{3.4}$$

$$\frac{2\beta-2m-1}{\beta q} + n_0 - \frac{2}{q} = n_0 - 1, \tag{3.5}$$

and

$$\frac{\beta-2m-1}{\beta q} + n_0 - \frac{1}{q} = n_0 - 1. \tag{3.6}$$

From (3.4) and (3.5) we find that

$$\beta = (\alpha+m+1)/2, \tag{3.7}$$

and

$$q = 2(2m+1)/(\alpha+m+1), \tag{3.8}$$

and these values also satisfy equation (3.6).

Suppose now that there is a continuous point source, at the origin of coordinates, emitting Q grammes per second of air-borne particles of matter. If the strength of the source is constant, then the flux of matter across an infinite vertical plane must be equal to Q at all distances down-wind.

Hence χ must satisfy the following conditions:

$$\text{(i)} \quad \int_{-\infty}^{\infty}\int_{0}^{\infty} u(z)\chi(x,y,z)\,dydz = Q \text{ for all values of } x; \tag{3.9}$$

$$\text{(ii)} \quad \chi \to \infty, \qquad \xi \to \eta \to \zeta \to 0; \qquad \chi \to 0, \qquad \xi \to 0, \quad \eta > 0, \quad \zeta > 0. \tag{3.10}$$

Now χ is a function of the form

$$\chi = Cx^{n_0}\psi(u,v),$$

where C is a constant, determined by equation (3.9). By using (2.2) and (3.1), equation (3.9) becomes in (u,v) coordinates,

$$\frac{CU_1}{\beta z_1^m} x^{[n_0+1/q+(m+1)/\beta q]} \int_{0}^{\infty}\int_{-\infty}^{\infty} \psi(u,v)v^{(m+1-\beta)/\beta}\,dudv = Q. \tag{3.11}$$

The coefficient of x is to be zero, and hence

$$n_0 = -[1+(m+1)/\beta]/q. \tag{3.12}$$

Substitution of the values of β and q given in equations (3.7) and (3.8) leads to

$$n_0 = -(\alpha+3m+3)/(4m+2),$$

or

$$\alpha = -2(2m+1)n_0-3m-3. \tag{3.13}$$

At this stage, if a satisfactory *experimental* law was known giving the variation of peak concentration (i.e. along the x-axis) with distance from a continuous point source for various degrees of atmospheric turbulence, such a law would enable n_0 to be determined in terms of m, and α follows from equation (3.13). Unfortunately, as indicated by O. G. Sutton (**5**), reliable data on diffusion over the whole range of atmospheric stabilities is not available. However, as mentioned by O. G. Sutton in this paper, it has been established *experimentally*, in adiabatic conditions, that the central or peak concentration from a continuous point source decreases with distance down-wind (x) according to the law $\chi \propto x^{-1\cdot76}$, i.e. $n_0 = -1\cdot76$.

Substituting this particular value for n_0 into equation (3.13) and taking the value of m as $\frac{1}{7}$ leads to $\alpha = 1\cdot09$, in the particular case of adiabatic conditions; hence in these conditions K_y appears to increase with height at a rate slightly greater than the linear law.

Although it is not possible to obtain an exact solution to the problem of the determination of α to cover the whole range of atmospheric stabilities, an approximate theory may be developed. O. G. Sutton (**5**) gives a continuous point-source formula based on his 'statistical' theory and from this work the law of variation of peak concentration is written $x^{-(2-n)}$. The index has been fully verified in the neighbourhood of adiabatic conditions. As indicated by Sutton, this theory is based on the assumption

that wind-speed variation with height may be neglected and this leads to only a small error in practice. However, if comparison is to be made with the law $x^{-(2-n)}$, the basic equation (2.1) must be treated with $m = 0$ in the term $U(\partial\chi/\partial x)$, i.e. the assumption must be made that wind-speed variation with height may be neglected.

We therefore begin by writing equation (2.5) in the new form

$$\frac{\partial\chi}{\partial x} = A\frac{\partial}{\partial z}\left(z^{1-m}\frac{\partial\chi}{\partial z}\right)+B\frac{\partial}{\partial y}\left(z^{\alpha}\frac{\partial\chi}{\partial y}\right). \tag{3.14}$$

The treatment proceeds precisely as detailed above, but from this new equation. The value of β is found to be given again by equation (3.7), i.e.

$$\beta = (\alpha+m+1)/2,$$

and the new expression for q is

$$q = 2(m+1)/(\alpha+m+1). \tag{3.15}$$

When the variation of wind speed with height is neglected, equation (3.12) for n_0 becomes

$$n_0 = -(1+1/\beta)/q.$$

Substitution of equations (3.7) and (3.15) into this formula for n_0 leads to

$$n_0 = -(\alpha+m+3)/2(m+1),$$

or

$$\alpha = -2(m+1)n_0-m-3. \tag{3.16}$$

The relation $n_0 = -(2-n)$ is given by O. G. Sutton's statistical theory, and substitution in equation (3.16) gives $\alpha = 1-m$. On the basis of this approximate theory the empirical power law of variation of K_y with height therefore appears to be

$$K_y = a_y\, U_1^{1-n}z^{1-m},$$

which is the same law as that used for K_z.

In the actual flow of air over a rough surface the physical conditions are, of course, very complex and probably different laws of variation of wind speed and diffusivities hold in different height ranges, especially near the ground. The best that we can hope to achieve with a mathematical theory at the moment is to assume that our model of diffusion is true over the whole height range where diffusion is expected to take place down to $z = 0$. Complete agreement with experiment justifies this step in two-dimensional problems; in three dimensions agreement is necessarily obtained for peak concentration variation, but a complete test awaits further theoretical work on cloud widths from a continuous point source. When the appropriate values of α, q, and n_0 are substituted into equation (3.3), it leads to an equation, in u, v coordinates, which is not separable, and it appears that the full solution for a point source requires a numerical evaluation of the ψ function. It is only in the particular case of $\alpha = m$ that the equation is

separable and the continuous point source solution may be obtained in an explicit mathematical form, as given in Appendix I. The application of the case $\alpha = m$ to evaporation problems is discussed in the following section.

4. The evaporation problem: use of $K_y = a_y U_1^{1-n} z^m$

It is clear that the two-dimensional evaporation theory worked out by O. G. Sutton (**3**) and K. L. Calder (**4**) is only applicable if the cross-wind width of the evaporating area is large, except at points on the central axis of the area, not too far down-wind. At points just outside the area contained between the lines running directly down-wind, from the longitudinal edges of a rectangular evaporating area, for example, the two-dimensional theory obviously gives zero concentration of vapour, whereas it is well known from experiment that a considerable concentration is in fact produced. This is due to the fact that near the surface the lateral eddies have a greater energy than the vertical components.

The author has succeeded in solving the problem of evaporation and vapour distribution over and down-wind of areas of parabolic (**1**) and rectangular shape (**2**), using a model of turbulence given by $K_y = a_y U_1^{1-n} z^m$. The parabolic solution was expressed in terms of easily computed functions and the rectangular solution in terms of Mathieu functions which are not yet completely tabulated. From the table in Appendix 2 it is seen that for a continuous point source solution the experimental variation of peak concentration down wind follows the law $x^{-1\cdot76}$, whereas in the case K_y proportional to z^m it is $x^{-1\cdot40}$; both in adiabatic conditions. For large values of x the error introduced by using the z^m law is clearly very considerable. However, an evaporating area may for mathematical convenience be considered to be equivalent to a very close grid of point sources, and at a point over the area the major portion of the concentration is due to those sources *near* the point under consideration and hence the previous solution for the vapour distribution over parabolic and rectangular areas may be expected to lie very near the truth. This also applies to distances not too far down-wind: the actual limit may only be determined by experiment. In support of this expectation the great improvement in the theoretical vapour distribution over a parabolic area, indicated by the author (**1**), must be quoted. In addition the results of experiments in the field, arranged by the Ministry of Supply, over the down-wind edge of a contaminated parabolic strip of short grassland, 10 yards long, indicate as closely as can be expected in field trials of this type that the 'z^m law' produces a *fully sufficient* degree of lateral diffusion: this effect was *particularly noticeable at the outside boundary of the area*, i.e.

at a distance of 5 *yards* from the axis of the area. The results of a typical experiment are given in Table 1. The method used for the theoretical computation has been described previously by the author (**1**). Direct comparison between theory and experiment is not possible since theory gives the concentration $\chi(x, y, z)$ at any point above the area in terms of χ_0, the ground concentration on the evaporating area. As χ_0 is not known,

TABLE 1 (*a*)

Lateral distance from the axis	*Type of result*	*Height of sampling point*			
		1″	8″	24″	48″
0 yd.	Experimental	1·00	0·35	0·06	0·01
	Theoretical, based on $K_y \propto z^m$	1·00	0·38	0·11	0·02
	Theoretical, based on $K_y = 0$	1·00	0·46	0·16	0·04
2·5 yd.	Experimental	0·68	0·26	0·04	0·004
	Theoretical, based on $K_y \propto z^m$	0·95	0·33	0·08	0·007
	Theoretical, based on $K_y = 0$	0·95	0·39	0·11	0·02
5·0 yd.	Experimental	0·15	0·05	0·01	0·003
	Theoretical, based on $K_y \propto z^m$	0·37	0·09	0·02	0·003
	Theoretical, based on $K_y = 0$	0	0	0	0
7·5 yd.	Experimental	0·01	0·004	0·003	0·002
	Theoretical, based on $K_y \propto z^m$	0·003	0·002	0·001	0
	Theoretical, based on $K_y = 0$	0	0	0	0

Theoretical and experimental results giving mean values of relative concentrations at various points in a vertical plane above the down-wind edge of a parabolic evaporating area, initially contaminated with aniline to a density of 10 g. m.$^{-2}$ The area was 10 yards in length and 10 yards in width across the down-wind edge, with axis directed along mean air-stream, and vertex pointing up-stream.

TABLE 1 (*b*)

Wind speed at 2 metres height in cm./sec.	$\frac{\lvert\bar{v}'\rvert}{\lvert\bar{u}\rvert}$	$\frac{\lvert\bar{w}'\rvert}{\lvert\bar{u}\rvert}$	n
570	0·607	0·323	0·257

Observed meteorological data, measured over the period of the experiment.

comparison is made by expressing both theoretical and observed concentrations in terms of the concentration at a height of 1 in. at the centre of the down-wind edge of the area.

If the vapour distribution needs to be calculated at large distances down wind of the area, then the solution based on the law making K_y proportional to z^m will enable the distribution to be evaluated across the down-wind edge of the area—the first necessary step—and the method indicated in section 5 of the author's paper on the rectangle problem then adapted for

the more exact K_y law, using the substitutions **(3.1)** and **(3.2)** of the present paper. The problem, however, of evaluating numerically the function $\psi(u, v)$ will first have to be solved.

5. Discussion

The dependence of the theoretical variation of peak concentration from a continuous point source with distance has been worked out; and an empirical law of variation, with height, for the *lateral* coefficient of eddy diffusivity chosen to give agreement with well-known results. This new law differs from the law previously assumed (for mathematical convenience) in three-dimensional evaporation and diffusion problems. It appears that the previous law yields a point source solution providing insufficient diffusion, in the direction of the mean wind, at large distances from the source; but at small distances the error is small. Experimental results are quoted which indicate that the previous law gives good agreement for vapour diffusion at heights *over* an evaporating surface. This agreement is likely to continue until the diffusion at large distances downwind of the area is to be considered. For the solution to this particular problem and for a full solution to the continuous point source problem, the function ψ, defined in the present paper as a particular solution of a second-order partial differential equation in two variables, needs to be evaluated numerically.

I wish to express my thanks to Professor L. Rosenhead, F.R.S., for his interest in and criticism of the work. Acknowledgement is made to the Chief Scientist, Ministry of Supply, for permission to publish the contents of Table 1.

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APPENDIX I

Continuous point source solution in the case K_y proportional to z^m

The equation (2.5) may be simplified by means of the following transformations:

$$\xi = x, \qquad \eta = \left(\frac{U_1^n}{a_y z_1^m}\right)^{\frac{1}{2}} y,$$

$$\zeta = \frac{2}{2m+1}\left(\frac{U_1^n}{a_z z_1^m}\right)^{\frac{1}{2}} z^{m+\frac{1}{2}}. \tag{1}$$

The fundamental equation becomes

$$\zeta^s \frac{\partial \chi}{\partial \xi} = \frac{\partial}{\partial \eta}\left[\zeta^s \frac{\partial \chi}{\partial \eta}\right] + \frac{\partial}{\partial \zeta}\left[\zeta^s \frac{\partial \chi}{\partial \zeta}\right], \tag{2}$$

where $s = (2-n)/(2+n)$.

Now write
$$\frac{\eta}{\sqrt{\xi}} = u, \qquad \frac{\zeta}{\sqrt{\xi}} = v, \tag{3}$$
and
$$\chi = \xi^{n_0}\psi(u,v). \tag{4}$$

Substitution of (3) into (2) leads to the equation

$$n_0\psi - \frac{1}{2}\left(u\frac{\partial \psi}{\partial u} + v\frac{\partial \psi}{\partial v}\right) = \frac{\partial^2 \psi}{\partial u^2} + \frac{1}{v^s}\frac{\partial}{\partial v}\left(v^s \frac{\partial \psi}{\partial v}\right). \tag{5}$$

To solve, put $\psi = Y(u)Z(v)$, then

$$\frac{Y''}{Y} + \frac{u}{2}\frac{Y'}{Y} - n_0 = -\left\{\frac{(1/v^s)(d/dv)(v^s Z') + \frac{1}{2}vZ'}{Z}\right\} = -\alpha.$$

Hence it follows that
$$Y'' + \tfrac{1}{2}uY' = -\beta Y, \tag{6}$$
and
$$\frac{1}{v^s}\frac{d}{dv}(v^s Z') + \tfrac{1}{2}vZ' = \alpha Z, \tag{7}$$
where
$$\alpha - \beta = n_0.$$

In the above (ξ, η, ζ) coordinates, the condition (3.10) becomes

$$K\int_{-\infty}^{\infty}\int_{0}^{\infty} \chi \zeta^{1/(2m+1)}\, d\eta d\zeta = Q, \tag{8}$$

where
$$K = \frac{2}{(2m+1)}\frac{U_1}{z_1^m}\left(\frac{a_y z_1^m}{U_1^n}\right)^{\frac{1}{2}}\left[\left(\frac{2m+1}{2}\right)^{m+1}\left(\frac{a_z z_1^m}{U_1^n}\right)^{(m+1)/2}\right]^{2/(2m+1)}.$$

A suitable solution of (6) is
$$Y = e^{-u^2/4} = e^{-\eta^2/4\xi}, \tag{9}$$
when $\beta = +\frac{1}{2}$.

The equation for the Z function then becomes

$$\frac{1}{v^s}\frac{d}{dv}(v^s Z') + \tfrac{1}{2}vZ' = (\tfrac{1}{2} + n_0)Z. \tag{10}$$

Using equation (3), equation (8) becomes

$$K\xi^{n_0}\int_{-\infty}^{\infty}\int_{0}^{\infty} \psi\left(\frac{\eta}{\sqrt{\xi}}, \frac{\zeta}{\sqrt{\xi}}\right)\zeta^{1/(2m+1)}\, d\eta d\zeta = Q. \tag{11}$$

Now use (3) in (11) and it leads to

$$K\xi^{[n_0+1+1/(2m+1)]}\int_{-\infty}^{\infty}\int_{0}^{\infty} \psi(u,v)v^{1/(2m+1)}\, dudv = Q.$$

Hence
$$n_0 = -(1+\tfrac{1}{2}s_0), \tag{12}$$
where $s_0 = 1/(2m+1)$. Equation (10) becomes

$$Z'' + \frac{s_0}{v}Z' + \tfrac{1}{2}vZ' = -\tfrac{1}{2}Z - \tfrac{1}{2}s_0 Z.$$

This equation has the particular solution

$$Z = e^{-v^2/4} = e^{-\zeta^2/4\xi}. \tag{13}$$

The solution in (ξ, η, ζ) coordinates is thus of the form

$$\chi = \frac{A}{\xi^{1+s_0/2}} \exp\{-(\eta^2+\zeta^2)/4\xi\}, \tag{14}$$

where A is a constant easily determined from (8). Equation (14) satisfies conditions (3.10).

It may be seen that this solution for the case $\alpha = m$ may be arrived at immediately by assuming a trial solution of the form

$$\chi = Ax^{-p} \exp\{-(y^q+z^s)/x\}; \tag{15}$$

by substituting in equation (2.5) and comparing the indices of separate terms, the appropriate values of p, q, s may be determined. However, although the method outlined above is rather involved in comparison with the trial method, it may indicate the general lines along which the more difficult solution may proceed for the case when K_y is taken to be proportional to z^{1-m}.

APPENDIX II

Peak Concentration Variations (from Continuous Point Source) with Distance Down-wind, using Different Theoretical Lateral Diffusivities: Adiabatic Conditions assumed

Variations of			*Index*
K_y	K_z	U	$-n_0$
constant	constant	constant	1·00
z^m	z^m	z^m	1·07
z^m	z^{1-m}	z^m	1·40
z^{1-4m}	z^{1-m}	z^m	1·50
z^{1-m}	z^{1-m}	z^m	1·67
$z^{1+m(1-3m)/(1+m)}$	z^{1-m}	z^m	1·75

Experimental value of $-n_0 = 1{\cdot}76$.

NOTE ON THE CIRCULATORY FLOW ABOUT A CIRCULAR CYLINDER THROUGH WHICH THE NORMAL VELOCITY IS LARGE

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SUMMARY

In this paper we consider the circulatory viscous flow about a circular cylinder of infinite length, the porous surface of which allows a normal velocity of fluid which, at any instant, is constant at all points of the surface. Asymptotic series in inverse powers of a suction Reynolds number are obtained, in steady and unsteady flows, for the distributions of circulation about circles concentric with the cylinder, pressure, and torque on the cylinder. Various comparisons are made with conventional boundary-layer theory.

1. Introduction

THE properties of circulation about two-dimensional aerofoils and other bodies have received a certain amount of attention recently. In particular, it has been suggested (**1**) that it is possible to set up circulation about any suitably rounded surface in a uniform stream and this has been achieved experimentally in the case of a porous circular cylinder fitted with a flap. Whether or not, on removal of the flap, the circulation at infinity remains constant or decreases is important, and experiments are being designed to investigate the matter. These experiments will include attempts to obtain circulation about a fixed cylinder in otherwise still air.

The flow considered in this paper is the radial symmetrical flow outside a fixed porous circular cylinder on which is superposed a distribution of circulation. Exact solutions have been given for such a steady flow (**2**) which show, *inter alia*, that the suction velocity must exceed a certain minimum value for a non-zero value of circulation at infinity: for otherwise the vorticity will diffuse sufficiently rapidly through the fluid to destroy this circulation. No general solution in finite terms exists for unsteady flow.

The principal object of this paper is to find general solutions of flow in the form of asymptotic series in a parameter involving the suction velocity from which the velocity and pressure fields and torque on the cylinder can be calculated in cases where no general solution exists. These solutions not only yield interesting analogies to conventional boundary-layer theory (**3**) but also will provide direct comparisons with experiment. It is shown that, for values of the suction velocity appropriate to this asymptotic theory, the circulation at infinity remains constant.

2. Notation and equations

Cylindrical polar coordinates (r, ϕ, z) are used, with the corresponding velocity components (v, u, w). If derivatives with respect to ϕ and z are identically zero and $w = 0$, the equation of continuity gives $\frac{\partial}{\partial r}(vr) = 0$ or $vr = \text{constant} = +v_0 a$, $v_0 = v_0(t)$ being the normal velocity at the cylinder $r = a$. The ϕ-equation of motion is

$$\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial r}+\frac{uv}{r} = \nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right),$$

where ν is the kinematic viscosity.

If the circulation $k = 2\pi ur$ is taken as the dependent variable and the value given above for v is used, this equation becomes

$$r\frac{\partial^2 k}{\partial r^2}+(R-1)\frac{\partial k}{\partial r}-\frac{r}{\nu}\frac{\partial k}{\partial t} = 0 \tag{1}$$

in which R is the Reynolds number $-\frac{v_0 a}{\nu}$. If the cylinder is at rest, the boundary conditions are $k = 0$ on $r = a$, and $k \to k_0(t)$ as $r \to \infty$, $k_0(t)$ being the arbitrary time-distribution of circulation at infinity. The r-equation of motion reduces to

$$\nu r^2\frac{\partial R}{\partial t}+v_0^2 a^2+\frac{k^2}{4\pi^2} = +\frac{r^3}{\rho}\frac{\partial p}{\partial r}, \tag{2}$$

where ρ is the density and p is the pressure of the fluid. This gives the distribution of pressure. Also $p \to p_0$ as $r \to \infty$. The torque M per unit span exerted by the fluid on the cylinder is given simply by

$$M = \mu a\left(\frac{\partial k}{\partial r}\right)_{r=a} \tag{3}$$

where μ is the coefficient of viscosity.

3. Steady motion

(i) When the flow is independent of time, t, solutions of (1), (2), and (3) are given in (2) and are

$$\left.\begin{aligned} k &= k_0\left\{1-\left(\frac{a}{r}\right)^{R-2}\right\} \\ p_0-p &= \frac{1}{8\pi^2}\left(\frac{k_0^2\rho}{a^2}\right)\left(\frac{a}{r}\right)^2\left[1-\frac{4}{R}\left(\frac{a}{r}\right)^{R-2}+\frac{1}{R-1}\left(\frac{a}{r}\right)^{2(R-2)}+\frac{4\pi^2 v_0 a^2}{k_0^2}\right] \\ M &= k_0\mu(R-2) \end{aligned}\right\}. \tag{4}$$

These solutions show that for conditions at infinity to be satisfied we must have $R > 2$, or that steady motion is impossible when $-v_0 \leqslant (2\nu/a)$. They

also show that the circulation tends to its value k_0 at infinity slowly, and in particular more slowly than the exponential manner in which the velocity in a boundary-layer on a plane surface tends to its stream velocity. However, it can be seen that the circulation increases rapidly near the surface of the cylinder, the rate of increase increasing with R. But these solutions, as they stand, give no indication of particular behaviour when R is very large or of the limit of the circulation distribution as $R \to \infty$. However, these conditions may be approximated to in experiment and it is the purpose of this paper to display the behaviour of solutions in asymptotic series in R.

(ii) Now the circulation-loss layer, or the layer near the cylinder's surface in which the circulation increases rapidly to near its asymptotic value, varies with R and some assumption must be made about the variation. The vorticity which is diffused outwards from the surface is convected inward by the suction velocity, and so when the suction velocity is large, the total distance through which vorticity is transported outward from the cylinder is inversely proportional to the suction velocity. Thus, on the hypothesis that the circulation-loss layer is inversely proportional to the suction velocity, we put

$$r = a+\eta a/R, \tag{5}$$

η being a non-dimensional measure of radial distance y from the surface of the cylinder. Then $\eta = (-v_0 y)/\nu$ and is a familiar parameter of conventional boundary-layer suction theory. If this form (5) for r is inserted into the solutions (4), it is possible to expand each expression in a series in inverse powers of R by elementary means to obtain:

$$\left.\begin{aligned} k &= k_0\Big\{(1-e^{-\eta})-\frac{\eta}{2R}(\eta+4)e^{-\eta}-\frac{\eta^2}{24R^2}(3\eta^2+16\eta+24)e^{-\eta}- \\ &\qquad -\frac{\eta^4}{48R^3}(\eta^2+4\eta+4)e^{-\eta}+O\Big(\frac{1}{R^4}\Big)\Big\} \\ \frac{(p_0-p)r^2}{\frac{1}{2}\rho v_0^2 a^2} &= 1+\Big(\frac{k_0}{2\pi\nu}\Big)^2\Big\{\frac{1}{R^2}-\frac{(4e^{-\eta}-e^{-2\eta})}{R^3}- \\ &\qquad -\frac{(2\eta^2+8\eta)e^{-\eta}-(\eta^2+4\eta+1)e^{-2\eta}}{R^4}+O\Big(\frac{1}{R^5}\Big)\Big\} \\ \frac{M}{-\rho a k_0 v_0} &= 1-\frac{2}{R} \end{aligned}\right\}. \tag{6}$$

These are the required asymptotic expressions. The first series shows that as $R \to \infty$, the distribution of circulation (in circles concentric with the cylinder) is precisely similar to the distribution $u/U = 1-e^{v_0 y/\nu}$ which

gives the velocity u in a boundary-layer at a point where the stream velocity is U and the suction velocity v_0 is very large (3) and negative. The second series shows that the pressure differences in the field are proportional to v_0^2 and the last that the torque is proportional to v_0, a result which follows, of course, from the assumption that the circulation-loss layer thickness is inversely proportional to v_0.

(iii) The series in (6) can also be derived from the appropriate equations with η as an independent variable in place of r. For example, in the steady motion case under consideration, (1) becomes

$$\left(1+\frac{\eta}{R}\right)\frac{\partial^2 k}{\partial \eta^2}+\left(1-\frac{1}{R}\right)\frac{\partial k}{\partial \eta}=0,$$

and on the assumption that

$$\left.\begin{aligned} k &= k_0\left(f_0(\eta)+\frac{1}{R}f_1(\eta)+\frac{1}{R^2}f_2(\eta)+\dots\right), \\ f_0(0) &= f_n(0)=f_n(\infty)=0, \qquad n \geqslant 1, \qquad f_0(\infty)=1, \end{aligned}\right\} \tag{7}$$

successive equations are obtained for $f_0(\eta)$, $f_1(\eta)$ and so on, which can be solved to give the series already obtained in (6). This procedure therefore merely checks the simpler method of obtaining the asymptotic series from the exact results. However, in any case where an exact explicit solution is not known, this procedure is the only possible one, and it is now applied to the case of unsteady flow.

4. Unsteady motion

(i) If the circulation, k_0, at infinity is not constant, then it can be easily deduced from (1) by considerations of the behaviour of k as $r \to \infty$ that R must be less than 2; and when $R < 2$ it appears that the distribution of k_0 with t can be arbitrarily assigned.† This presents a difficulty in the determination of the decay of k_0 when initial conditions are postulated. However, an asymptotic investigation for large R is unlikely to be valid for $R < 2$ when in any case the transformation (5) to η is inappropriate. The case $R > 2$ only is therefore considered when k_0 must remain constant and R or the suction velocity varies with time. It may be remarked here that, from (1), the conditions at time t_1 depend only upon what has

† With $R = 1$ as a particular example, (1) becomes $\frac{\partial k}{\partial t} = \nu \frac{\partial^2 k}{\partial r^2}$. This equation occurs in the theory of heat conduction and also is of the form of the so-called outer solution of Karman and Millikan's method of dealing with boundary-layer flow on a plane boundary. The equation can be solved in terms of the variable $\frac{r-a}{2\sqrt{(t\nu)}}$ and in particular, a series solution is obtainable in the case $k_0(t) = \sum_0^\infty a_n t^n$.

happened for $t < t_1$: this corresponds to the conventional boundary-layer theory result that conditions at any point depend only on what has happened upstream of that point.

(ii) When $R > 2$, equation (1) possesses no general solutions of the simplicity of the steady solutions of § 3; investigation of the behaviour of the flow when R is large (and not constant) is not as simple as in § 3, therefore, and must proceed by the use of series such as (7). The asymptotic transformation

$$R = R(T) = \frac{-v_0 a}{\nu}, \qquad t = \frac{a^2}{\nu} T, \qquad r = a\left(1+\frac{\eta}{R}\right), \qquad S(T) = \frac{R'}{R} \tag{8}$$

applied to equation (1) gives

$$\left(1+\frac{\eta}{R}\right)\frac{\partial^2 k}{\partial \eta^2}+\left(1-\frac{1}{R}\right)\frac{\partial k}{\partial \eta}-\left(1+\frac{\eta}{R}\right)\frac{1}{R^2}\left[\frac{\partial k}{\partial T}+\eta S\frac{\partial k}{\partial \eta}\right] = 0. \tag{9}$$

If the series (7) is inserted into (9), the f's now being functions of η and t, the following equations are obtained:

$$\left.\begin{aligned}
&\frac{\partial^2 f_0}{\partial \eta^2}+\frac{\partial f_0}{\partial \eta} = 0\\
&\frac{\partial^2 f_1}{\partial \eta^2}+\frac{\partial f_1}{\partial \eta} = -\eta\frac{\partial^2 f_0}{\partial \eta^2}+\frac{\partial f_0}{\partial \eta}\\
&\frac{\partial^2 f_2}{\partial \eta^2}+\frac{\partial f_2}{\partial \eta} = -\eta\frac{\partial^2 f_1}{\partial \eta^2}+\frac{\partial f_1}{\partial \eta}+S\eta\frac{\partial f_0}{\partial \eta}+\frac{\partial f_0}{\partial T}\\
&\frac{\partial^2 f_3}{\partial \eta^2}+\frac{\partial^2 f_3}{\partial \eta} = -\eta\frac{\partial^2 f_2}{\partial \eta^2}+\frac{\partial f_2}{\partial \eta}+S\eta^2\frac{\partial f_0}{\partial \eta}+S\eta\frac{\partial f_1}{\partial \eta}-Sf_1+\eta\frac{\partial f_0}{\partial T}+\frac{\partial f_1}{\partial T}\\
&\frac{\partial^2 f_4}{\partial \eta^2}+\frac{\partial f_4}{\partial \eta} = -\eta\frac{\partial^2 f_3}{\partial \eta^2}+\frac{\partial f_3}{\partial \eta}+S\eta^2\frac{\partial f_1}{\partial \eta}+\\
&\qquad\qquad +S\eta\frac{\partial f_2}{\partial \eta}-S\eta f_1-2Sf_2+\eta\frac{\partial f_1}{\partial T}+\frac{\partial f_2}{\partial T}
\end{aligned}\right\} \tag{10}$$

and so on. These equations can be solved simply by separation of variables; f_0 and f_1 are functions of η only, and are given in the first two terms of the series for k in (6); f_2 and f_3 can be written as

$$f_2(\eta, T) = g_2(\eta)+Sh_2(\eta), \qquad f_3(\eta, T) = g_3(\eta)+Sh_4(\eta), \tag{11}$$

the g's and h's being functions of η alone. For example, the equation for h_2 is

$$\frac{\partial^2 h_2}{\partial \eta^2}+\frac{\partial h_2}{\partial \eta} = \eta\frac{\partial f_0}{\partial \eta}, \qquad h_2(0) = h_2(\infty) = 0. \tag{12}$$

The particular integral of f_4 clearly involves S, S', and S^2, and so we put

$$f_4(\eta, T) = g_4(\eta)+Sh_4(\eta)+S^2 j_4(\eta)+S'k_4(\eta), \tag{13}$$

and f_n, $n > 4$, can be expressed similarly in terms of combinations of S and its derivatives. The g's are given already in (6) (the steady case in which $S \equiv 0$).

To the series in (6) for k must be added the following terms:

$$-\frac{\frac{1}{2}\eta(\eta+2)e^{-\eta}S}{R^2}-\frac{1}{24}\frac{\eta(3\eta^3+32\eta^2+96\eta+192)e^{-\eta}S}{R^3}+O\left(\frac{1}{R^4}\right), \tag{14}$$

and so, from (3), to the series for $M/(-\rho a k_0 v_0)$ the terms

$$-\frac{S}{R^2}-\frac{8S}{R^3}+O\left(\frac{1}{R^4}\right). \tag{15}$$

The effect of variation of R on pressure p is found from (2). First there is the addition to p due to the term $\nu r^2 \dfrac{\partial R}{\partial T}$ which is easily calculable and means the replacement of $\dfrac{(p_0-p)r^2}{\frac{1}{2}\rho v_0^2 a^2}$ in (6) by

$$\frac{\{p_0-p+(\rho\nu^2/a^2)SR\log(r/a)\}r^2}{\frac{1}{2}\rho v_0^2 a^2}.$$

Then there is the effect of the additional terms (14) which, however, only occurs for terms of $O(R^{-5})$ in the series (6).

The asymptotic series have now been found for the case of unsteady flow in which no simple exact solutions can be found. A feature of this asymptotic solution is that conditions at time t depend only upon the value of R and of v_0 and its time-derivatives at that time and not upon previous happenings. This corresponds to the result of steady boundary-layer theory that conditions at a point where the suction velocity is large depend only upon the values at that point of the suction velocity, the stream velocity and their derivatives along the surface. The present results therefore not only will provide a valuable check on coming experimental work but also provide interesting comparisons with conventional boundary-layer theory.

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THE EQUILIBRIUM DISTRIBUTION OF THE LONG-PERIOD TIDES OVER AN OCEAN COVERING THE NORTHERN HEMISPHERE

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SUMMARY

Results of a calculation of the equilibrium distribution of the long-period tides over an ocean covering the northern hemisphere of a uniform spherical globe are given here making allowance for the change in the gravitational field produced by the tides themselves. The calculation is based on a normalization process described by M. Brillouin in 1927. The main result shows that, on the assumptions on which Brillouin's paper is based, making allowance for the mutual attraction of the elevated water gives an elevation which approximates very closely to that which would be obtained by multiplying the uncorrected level by the constant 1·126.

1. Introduction

By the equilibrium distribution of tides is meant the distribution as calculated from the tide-generating forces of the moon or sun on statical principles. The problem of making such a calculation over an ocean bounded by continents is simple so long as the mutual attraction of the elevated water is neglected. The solution is given in Lamb's *Hydrodynamics* (**1**). If there are no continents, so that the ocean covers the whole globe, then the problem of allowing for the mutual attraction of the elevated water is also simple, and the solution is given by Thomson and Tait (**2**). But when both continents and elevated water are taken into account, the problem is difficult, and no case has previously been worked out in detail. A general analysis of the problem was given by H. Poincaré (**3**) in 1895, and another by M. Brillouin (**4**) in 1927. The present paper consists of an application of Brillouin's method to the long-period tides in an ocean covering the northern hemisphere.

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2. Notation and introductory theory

(r, θ, ϕ), spherical polar coordinates with origin at the centre of the earth which is assumed spherical; r denoting the radius, θ the co-latitude, and ϕ the longitude.

a, the mean radius of the earth.

γ, the constant of gravitation.

g, the uniform acceleration due to the attraction of the earth.

ρ, the mean density of the ocean ($= 1{\cdot}026$ gm./c.c.).

$\bar{\rho}$, the mean density of the earth ($= 5{\cdot}527$ gm./c.c.).

$k = (4\pi\gamma\rho a)/g = 3\rho/\bar{\rho} = 0{\cdot}5569$.

S_n, the spherical surface harmonic of integral order t where

$$t^2 \leqslant n < (t+1)^2.$$

$P_n = P_n(\cos\theta)$, the Legendre polynomial of integral order n.

$V_a = gH(P_2+D)$, where gHP_2 is the variable part of the Newtonian potential at the earth's mean surface due to the disturbing body and the earth's gravitation, H being an appropriate length, and D is a numerical constant.

V', the Newtonian potential due to the elevated water.

ζ, the elevation of the free surface of the ocean measured radially.

The elevation of the free surface of the ocean is given by the equation

$$g\zeta = V_a + V'_a,$$

where V_a is known except for an arbitrary constant and V'_a, the value of V' when $r = a$, is to be calculated.

Brillouin showed that

$$V' = \sum A_n \Theta_n \quad \text{or} \quad \sum A_n \Phi_n,$$

where

$$C_{0,0}\Theta_0 = \frac{a}{r} S_0,$$

. . . .

$$C_{n,n}\Theta_n = C_{0,n}\Theta_0 + C_{1,n}\Theta_1 + \ldots + C_{n-1,n}\Theta_{n-1} + \left(\frac{a}{r}\right)^{t+1} S_n \quad (r \geqslant a);$$

$$C_{0,0}\Phi_0 = S_0,$$

. . . .

$$C_{n,n}\Phi_n = C_{0,n}\Phi_0 + C_{1,n}\Phi_1 + \ldots + C_{n-1,n}\Phi_{n-1} + \left(\frac{r}{a}\right)^{t} S_n \quad (r \leqslant a).$$

Here $C_{n,n}$ and $C_{m,n}$ are constants chosen so that $C_{n,n}$ is positive,

$$(C_{0,n})^2 + (C_{1,n})^2 + \ldots + (C_{n,n})^2 = \iint \left(\frac{2t+1-k}{a}\right)^2 S_n^2 \, ds,$$

$$C_{m,n} = \frac{1}{a} \iint \frac{2t+1-k}{a} \Psi_m S_n \, ds \quad (m < n),$$

$$\frac{1}{a}\Psi_n = \left[\frac{\partial}{\partial r}\Theta_n - \frac{\partial}{\partial r}\Phi_n + \frac{k}{a}\Phi_n\right]_{r=a}.$$

Further

$$A_n = -\frac{1}{a^2} \iint k V_a \Psi_n \, ds,$$

where ds $(= a^2 \sin\theta\, d\theta d\phi)$ is an element of the spherical surface and all the integrals are taken over the whole sphere.

The value of the constant D in V_a is calculated by using the fact that

$$\iint \zeta\, ds = 0,$$

where the integration extends over the ocean.

3. Long-period tides in an ocean covering the northern hemisphere

For an ocean bounded by the equator,

$$\begin{aligned} k &= (4\pi\gamma\rho a)/g = 3\rho/\bar{\rho} \quad \text{if} \quad 0 \leqslant \theta \leqslant \pi/2, \\ &= 0 \quad\quad\quad\quad\quad\quad\quad\quad \text{if} \quad \pi/2 < \theta \leqslant \pi. \end{aligned}$$

The effective potential producing the long-period tides does not depend on ϕ, so the coefficients A_n of § 2 are zero except where n is a perfect square, that is when S_n is a zonal harmonic of order $\sqrt{n}$. Consequently, S_n may be replaced by $P_n(\cos\theta)$ and t may be replaced by n everywhere. It follows that

$$\Psi_n = \frac{1}{C_{n,n}}[C_{0,n}\Psi_0 + C_{1,n}\Psi_1 + \ldots + C_{n-1,n}\Psi_{n-1} + (k - \overline{2n+1})P_n],$$

and on substituting for Ψ_r $(r = 1, 2, \ldots, n-1)$ and rearranging

$$\begin{aligned} \Psi_n &= \frac{1}{C_{n,n}}[b_{0,n}(k-1)P_0 + b_{1,n}(k-3)P_1 + \ldots + b_{n,n}(k - \overline{2n+1})P_n] \\ &= a_{0,n}(k-1)P_0 + a_{1,n}(k-3)P_1 + \ldots + a_{n,n}(k - \overline{2n+1})P_n, \end{aligned}$$

where the $a_{r,n}$ are constants and $C_{n,n}$, $C_{m,n}$ are now calculated from

$$(C_{0,n})^2 + (C_{1,n})^2 + \ldots + (C_{n,n})^2 = \iint \left(\frac{2n+1-k}{a}\right)^2 P_n^2\, ds$$

and

$$C_{m,n} = \frac{1}{a}\iint \frac{2n+1-k}{a}\Psi_m P_n\, ds \quad (m < n).$$

Since

$$\int_0^{\frac{1}{2}\pi} P_n^2 a^2 \sin\theta\, d\theta = \int_{\frac{1}{2}\pi}^{\pi} P_n^2 a^2 \sin\theta\, d\theta$$

it follows that

$$\begin{aligned} &\int_0^{2\pi}\int_0^{\pi} \left(\frac{2n+1-k}{a}\right)^2 P_n^2 a^2 \sin\theta\, d\theta d\phi \\ &\quad = 2\pi[(2n+1-k)^2 + (2n+1)^2]\int_0^{\frac{1}{2}\pi} P_n^2 \sin\theta\, d\theta \\ &\quad = 2\pi[2(2n+1) - 2k + k^2/(2n+1)], \end{aligned}$$

and so

$$(C_{0,n})^2 + (C_{1,n})^2 + \ldots + (C_{n,n})^2 = 2\pi[2(2n+1) - 2k + k^2/(2n+1)]. \tag{1}$$

Since $\int\limits_0^{\frac{1}{2}\pi} P_s P_n \sin\theta\, d\theta = 0$ if s and n are both even or both odd,

$$= \frac{(-1)(-\frac{1}{2})^{(s+n-1)/2}1.3....(s-1).1.3....n}{(s-n)(s+n+1).(s/2)!\,[(n-1)/2]!} \text{ if } s \text{ is even and } n \text{ is odd,}$$

and
$$\int\limits_0^{\frac{1}{2}\pi} P_s P_n \sin\theta\, d\theta = -\int\limits_{\frac{1}{2}\pi}^{\pi} P_s P_n \sin\theta\, d\theta,$$

it follows that

$$\left.\begin{aligned} &C_{m,n} \\ &= 2\pi \int\limits_0^{\pi} (2n+1-k)P_n\Big[\sum_{s=0}^{m} a_{s,m}(k-\overline{2s+1})P_s\Big]\sin\theta\, d\theta \quad (m<n) \\ &= 2\pi \sum_{s=0}^{m} k[2(n+s+1)-k]\frac{a_{s,m}(-1)(-\frac{1}{2})^{(s+n-1)/2}1.3....(s-1).1.3....n}{(s-n)(s+n+1)\,(s/2)!\,[(n-1)/2]!} \\ &\qquad \text{if } s \text{ is even and } n \text{ is odd;} \\ &= 2\pi \sum_{s=0}^{m} k[2(n+s+1)-k]\frac{a_{s,m}(-1)(-\frac{1}{2})^{(s+n-1)/2}1.3....s.1.3....(n-1)}{(n-s)(s+n+1)\,[(s-1)/2]!\,(n/2)!} \\ &\qquad \text{if } s \text{ is odd and } n \text{ is even.} \end{aligned}\right\} \quad (2)$$

The calculation consists of three parts which are here labelled the first, second, and third process respectively. In the first process, using the equations (1) and (2), the coefficients $C_{n,n}$ and $C_{m,n}$, and the functions Ψ_n, are calculated as far as $n = 8$. They are calculated in the following order:

$$\begin{array}{llllll} C_{0,0}; & \Theta_0; & \Phi_0; & \Psi_0; & C_{0,n} & (0<n); \\ C_{1,1}; & \Theta_1; & \Phi_1; & \Psi_1; & C_{1,n} & (1<n); \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \quad \cdot \\ C_{m,m}; & \Theta_m; & \Phi_m; & \Psi_m; & C_{m,n} & (m<n) \end{array}$$

and the results are tabulated.

In the second process the coefficients A_n ($n = 0, 1,..., 8$) are calculated and tabulated, and the expression for the tidal elevation given in terms of Legendre polynomials and the undetermined constant D.

In the third process the constant D is evaluated by using the fact that the total amount of water remains constant, and the tidal elevation is calculated at co-latitudes 0°, 10°,..., 90°. For an ocean covering the whole globe the allowance for the mutual attraction of the elevated water can be made by multiplying the uncorrected level by the constant multiple 1·126 (**1**). The tidal elevation is now compared with the uncorrected level and with the uncorrected level multiplied by 1·126 at co-latitudes 0°, 10°,..., 90° and the results are tabulated.

The results of the calculation show that, for an ocean covering the northern hemisphere, a good approximation to the actual height is given by multiplying the uncorrected level by the constant 1·126.

First process

For convenience, the calculated expressions for Θ_n, Φ_n, and Ψ_n, for $n = 0, 1, \ldots, 8$, are collected together in Table 10. The appropriate values for $a_{s,m}$ in equation (2) can therefore be read off from this table remembering that $a_{s,m} = b_{s,m}/C_{m,m}$. Substituting those values in equation (2) gives $C_{m,n}$ for $m < n$. Equation (1) gives $C_{n,n}$ for $n = 0, 1, \ldots, 8$. The results are given in Tables 1 to 9.

TABLE 1

n	$C_{0,n}$	$(C_{0,n})^2$	$C_{0,n}/C_{0,0}$
0	2·741	7·515	1
1	2·198	4·828	0·8015
2	0	0	0
3	−1·188	1·410	−0·4331
4	0	0	0
5	0·9124	0·8326	0·3329
6	0	0	0
7	−0·7696	0·5924	−0·2807
8	0	0	0

TABLE 2

n	$C_{1,n}$	$(C_{1,n})^2$	$C_{1,n}/C_{1,1}$
1	5·148	26·512	1
2	0·6323	0·3997	0·1228
3	−0·5067	0·2568	−0·09842
4	−0·1619	0·02623	−0·03145
5	0·3894	0·1517	0·07563
6	0·08196	0·006717	0·01592
7	−0·3285	0·1079	−0·06380
8	−0·05160	0·002663	−0·01002

TABLE 3

n	$C_{2,n}$	$(C_{2,n})^2$	$C_{2,n}/C_{2,2}$
2	7·469	55·80	1
3	0·6268	0·3930	0·08395
4	−0·01371	0·0001879	−0·001835
5	−0·2495	0·06226	−0·03341
6	0·006937	0·00004812	0·0009287
7	0·1714	0·02938	0·02295
8	−0·004368	0·00001908	−0·0005849

TABLE 4

n	$C_{3,n}$	$(C_{3,n})^2$	$C_{3,n}/C_{3,3}$
3	8·898	79·17	1
4	0·4351	0·1893	0·04890
5	−0·1615	0·02607	−0·01814
6	−0·1236	0·01528	−0·01389
7	0·1335	0·01782	0·01501
8	0·06562	0·004305	0·007374

TABLE 5

n	$C_{4,n}$	$(C_{4,n})^2$	$C_{4,n}/C_{4,4}$
4	10·30	106·1	1
5	0·4517	0·2041	0·04386
6	−0·006520	0·00004250	−0·0006330
7	−0·1708	0·02916	−0·01658
8	0·003590	0·00001289	0·0003485

TABLE 6

n	$C_{5,n}$	$(C_{5,n})^2$	$C_{5,n}/C_{5,5}$
5	11·40	130·0	1
6	0·3550	0·1261	0·03115
7	−0·08523	0·007264	−0·007476
8	−0·1053	0·01109	−0·009234

TABLE 7

n	$C_{6,n}$	$(C_{6,n})^2$	$C_{6,n}/C_{6,6}$
6	12·50	156·4	1
7	0·3695	0·1366	0·02954
8	−0·003980	0·00001584	−0·0003183

TABLE 8

n	$C_{7,n}$	$(C_{7,n})^2$	$C_{7,n}/C_{7,7}$
7	13·45	180·8	1
8	0·3083	0·09506	0·02293

TABLE 9

$C_{8,8}$	$(C_{8,8})^2$
14·37	206·5

Table 10

n	$b_{0,n}$	$b_{1,n}$	$b_{2,n}$	$b_{3,n}$	$b_{4,n}$	$b_{5,n}$	$b_{6,n}$	$b_{7,n}$	$b_{8,n}$
0	1								
1	0·8015	1							
2	0·09842	0·1228	1						
3	−0·5037	−0·08812	0·08389	1					
4	−0·05002	−0·03598	0·002267	0·04890	1				
5	0·3972	0·07155	−0·03483	−0·01599	0·04384	1			
6	0·03226	0·01951	−0·001323	−0·01442	0·0007340	0·03115	1		
7	−0·3383	−0·06167	0·02439	0·01389	−0·01689	−0·006556	0·02954	1	
8	−0·02326	−0·01284	0·0009159	0·007862	−0·0004442	−0·009394	0·0003591	0·02293	1

Values of $C_{n,n}\,\Theta_n$, $C_{n,n}\,\Phi_n$ and $C_{n,n}\,\Psi^*_n$ $(n = 0, 1, \ldots, 8)$.

On $r = a$, $C_{n,n}\,\Theta_n = C_{n,n}\,\Phi_n = \sum_{r=0}^{n} b_{r,n}\,P_r$. $C_{n,n}\,\Psi^*_n = \sum_{r=0}^{n} b_{r,n}(k-\overline{2r+1})P_r$.

Second process

Table 11

n	A_n/gH	$A_n/gHC_{n,n}$
0	0 $+D(0·5654)$	0 $+D(0·2063)$
1	0·2075 $+D(1·072)$	0·04031 $+D(0·2082)$
2	0·4338 $+D(0·09067)$	0·05808 $+D(0·01214)$
3	0·3354 $+D(-0·4467)$	0·03769 $+D(-0·05020)$
4	0·01034 $+D(-0·03584)$	0·001004 $+D(-0·003480)$
5	−0·1319 $+D(0·2850)$	−0·01157 $+D(0·02500)$
6	−0·005469 $+D(0·01961)$	−0·0004375 $+D(0·001569)$
7	0·08652 $+D(-0·2093)$	0·006434 $+D(-0·01557)$
8	0·003480 $+D(-0·01251)$	0·0002422 $+D(-0·0008704)$

Values of A_n/gH and $A_n/gHC_{n,n}$ $(n = 0, 1, \ldots, 8)$.

$$A_n = -\iint \frac{kV_a}{a^2}\Psi^*_n\,ds, \quad V_a = gH(P_2+D).$$

$$\begin{aligned} g\zeta &= gH(P_2+D)+\sum A_n\,\Phi_n \qquad \text{(using Table 12)} \\ &= gH(P_2+D)+gH\{[0·01218+D(0·4150)]P_0 \\ &\quad +[0·04285+D(0·2170)]P_1+[0·06180+D(0·006668)]P_2 \\ &\quad +[0·03802+D(-0·05102)]P_3+[0·00039+D(-0·002119)]P_4 \\ &\quad +[-0·01163+D(0·02516)]P_5+[-0·00025+D(0·001109)]P_6 \\ &\quad +[0·006440+D(-0·01559)]P_7+[0·000242+D(-0·000870)]P_8\}. \end{aligned}$$

TABLE 12

n	$d_{0,n}$	$d_{1,n}$	$d_{2,n}$	$d_{3,n}$	$d_{4,n}$	$d_{5,n}$	$d_{6,n}$	$d_{7,n}$	$d_{8,n}$
0	0 $+D(0·2063)$								
1	0·03230 $+D(0·1668)$	0·04031 $+D(0·2082)$							
2	0·005716 $+D(0·001195)$	0·007132 $+D(0·001490)$	0·05808 $+D(0·01214)$						
3	−0·01899 $+D(0·02528)$	−0·003321 $+D(0·004424)$	0·003161 $+D(-0·004211)$	0·03769 $+D(-0·05020)$					
4	−0·000050 $+D(0·000174)$	−0·000036 $+D(0·000125)$	0·000002 $+D(-0·000008)$	0·000049 $+D(-0·000170)$	0·001004 $+D(-0·003480)$				
5	−0·004595 $+D(0·009931)$	−0·000828 $+D(0·001788)$	0·000403 $+D(-0·000871)$	0·000185 $+D(-0·000400)$	−0·000507 $+D(0·001096)$	−0·01157 $+D(0·02500)$			
6	−0·000014 $+D(0·000051)$	−0·000009 $+D(0·000031)$	0·000001 $+D(-0·000002)$	0·000001 $+D(-0·000023)$	−0·000000 $+D(0·000001)$	−0·000014 $+D(0·000049)$	−0·000437 $+D(0·001569)$		
7	−0·002177 $+D(0·005267)$	−0·000397 $+D(0·000960)$	0·000157 $+D(-0·000380)$	0·000089 $+D(-0·000216)$	−0·000109 $+D(0·000263)$	−0·000042 $+D(0·000102)$	0·000190 $+D(-0·000460)$	0·006434 $+D(-0·01557)$	
8	−0·000006 $+D(0·000020)$	−0·000003 $+D(0·000011)$	0·000000 $+D(-0·000001)$	0·000002 $+D(-0·000007)$	−0·000000 $+D(0·000000)$	−0·000002 $+D(0·000008)$	0·000000 $+D(-0·000000)$	0·000006 $+D(-0·000020)$	0·000242 $+D(-0·000870)$

Table giving $A_n \Phi_n / gH$ ($n = 0, 1, \ldots, 8$). $A_n \Phi_n / gH = d_{0,n} P_0 + \ldots + d_{n,n} P_n$

Third process

Since $\iint \zeta\, ds = 0$, where the integration is taken over the ocean,

$$D = -0{\cdot}01819.$$

$$\begin{aligned}\zeta/H = P_2 &- 0{\cdot}01819 + (0{\cdot}01218 - 0{\cdot}00755)P_0 \\ &+ (0{\cdot}04285 - 0{\cdot}00395)P_1 + (0{\cdot}06180 - 0{\cdot}00012)P_2 \\ &+ (0{\cdot}03802 + 0{\cdot}00093)P_3 + (0{\cdot}00039 + 0{\cdot}00004)P_4 \\ &+ (-0{\cdot}01163 - 0{\cdot}00046)P_5 + (-0{\cdot}00025 - 0{\cdot}00002)P_6 \\ &+ (0{\cdot}00644 + 0{\cdot}00028)P_7 + (0{\cdot}00024 + 0{\cdot}00002)P_8.\end{aligned}$$

TABLE 13

θ	0°	10°	20°	30°	40°	50°
ζ/H	1·121	1·069	0·920	0·696	0·423	0·132
ζ_1/H	1	0·955	0·824	0·625	0·380	0·120
ζ_2/H	1·126	1·075	0·928	0·704	0·428	0·135

θ	60°	70°	80°	90°
ζ/H	−0·144	−0·366	−0·504	−0·544
ζ_1/H	−0·125	−0·324	−0·455	−0·5
ζ_2/H	−0·141	−0·365	−0·512	−0·563

Table giving the tidal elevation at co-latitudes 0°, 10°, ..., 90°.

In these tables ζ denotes the tidal elevation when allowance is made for the attraction of the elevated water, ζ_1 denotes the uncorrected level, ζ_2 denotes the uncorrected level multiplied by 1·126

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THE APPLICATION OF BOUNDARY-LAYER THEORY TO SWIRLING LIQUID FLOW THROUGH A NOZZLE

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SUMMARY

The passage of a swirling liquid through a convergent-divergent nozzle is accompanied by the formation of a retarded layer on the nozzle wall. On the assumption that this layer is thin, the motion in the main stream may be taken as frictionless; and the pressure and velocity distributions in it, when the flow is derived from a high-pressure reservoir, can be obtained by an application of critical flow theory. As usual, it is necessary to suppose that the angle of the conical nozzle is small so that radial velocities can be disregarded. The streaming velocity is found to be uniform over each cross-section, and it is equal at the throat to the velocity of a 'long' wave moving on the surface of the air core.

With the aid of the results mentioned above, the boundary-layer equations are worked out by an extension of the method used by Taylor in his study of swirling flow without streaming, and a solution is obtained with the use of the Pohlhausen approximation. A numerical case is examined in detail, and the boundary layer is compared with those caused by swirl with no streaming and by streaming with no swirl. The effects of surface-tension over the air core are considered and are shown to be negligible in the numerical example.

1. Introduction

THE problem of swirling fluid motion inside a conical surface was attacked by Taylor (**1**). He considered a state in which the axial velocity in the main stream is negligible compared with the swirl; and, on applying the approximations of boundary-layer theory, he found that the retarded layer near the wall cannot maintain its position but is forced towards the apex of the cone. The results were pushed to a numerical conclusion, and in typical examples the boundary layer was shown to occupy a very large fraction of the cross-section at the outlet end of the conical frustum. Thus in an actual nozzle it may have an important influence upon the discharge. Taylor's work also explains the action of the 'Cyclone' method of dust extraction from air, which was tested by Linden (**2**). In this device the contaminated air is admitted tangentially to a conical vessel. The particles of dust are impelled by centrifugal force to the curved wall, along which they move in the boundary layer towards the closed apex end, where they are trapped. The cleaned air escapes on the axis at the other end of the vessel.

The swirling flow of a perfect liquid through a convergent-divergent nozzle was examined by Binnie and Hookings (**3**) and by Binnie (**4**), who obtained an expression for the discharge under given conditions. Making Reynolds's assumption that the radial velocities can be neglected, they proved that the axial velocity is uniform over the cross-section; with unimpeded flow, it attains at the throat a value equal to the velocity of a 'long' wave moving axially on the spinning surface of the core. Their analysis deals with motion under gravity, but it may easily be modified to apply to the case, now to be considered, of flow derived from a high-pressure reservoir.

In the following pages an attempt is made, in the manner described by Taylor, to derive the laminar boundary-layer equations when the main stream of liquid possesses both swirl and axial velocity, the latter being calculated by means of the approximate method mentioned in the previous paragraph. The equations are then solved in order to provide estimates of the thickness of the layer and of the discharge through it.

2. The motion in the main stream

We consider a nozzle discharging freely and supplied from a tank in which the absolute pressure is P, measured at a point where the velocity is negligible. The flow being irrotational, the swirl velocity V at radius r is given by an expression of the form

$$V = \Omega/r, \tag{2.1}$$

in which Ω is constant throughout the liquid. At a cross-section where the radii of the nozzle and of the core are a and b, it follows from Bernoulli's equation that if W and p are the streaming velocity and the absolute pressure in a liquid of density ρ,

$$\frac{p}{\rho}+\frac{1}{2}\left(W^2+\frac{\Omega^2}{r^2}\right) = \frac{P}{\rho}. \tag{2.2}$$

The radial velocity here has been omitted as being very small compared with the other two components. Now the variation of p over the cross-section is seen to be

$$\frac{p}{\rho} = \frac{\Omega^2}{2}\left(\frac{1}{b^2}-\frac{1}{r^2}\right), \tag{2.3}$$

hence from (2.2) it appears that W has the uniform value

$$W = \left(\frac{2P}{\rho}-\frac{\Omega^2}{b^2}\right)^{\frac{1}{2}}. \tag{2.4}$$

The discharge Q is then given by

$$Q = \pi(a^2-b^2)W. \tag{2.5}$$

The axial coordinate in the nozzle is denoted by z, measured positively in the direction of streaming; and on differentiating (2.4) and (2.5) with respect to z, and eliminating db/dz, we find that

$$\frac{\Omega^2 a}{b^4}\frac{da}{dz} = \frac{W^2 - W_1^2}{W}\frac{dW}{dz}, \tag{2.6}$$

where

$$W_1 = \left\{\frac{\Omega^2(a^2-b^2)}{2b^4}\right\}^{\frac{1}{2}} \tag{2.7}$$

was shown by Binnie and Hookings (**3**) to be the axial velocity of a 'long' wave on the free surface. Now at the throat, $da/dz = 0$ and $dW/dz \neq 0$, therefore at that cross-section $W = W_1$, and from (2.4) and (2.7) a quadratic equation for b^2 is obtained which yields

$$\frac{b_t^2}{a_t^2} = \frac{\rho\Omega^2}{8Pa_t^2}\left\{1+\left(1+\frac{16Pa_t^2}{\rho\Omega^2}\right)^{\frac{1}{2}}\right\}, \tag{2.8}$$

the suffix t referring to the throat. The other root, being negative, is irrelevant. From (2.5) and (2.7) we deduce, as the final expression for Q, that

$$\frac{Q}{\Omega a_t} = \frac{\pi}{\sqrt{2}}\frac{(1-b_t^2/a_t^2)^{\frac{3}{2}}}{b_t^2/a_t^2}, \tag{2.9}$$

and in this (2.8) is to be substituted. The non-dimensional quantities involved are shown in Fig. 1, where $Q/(\Omega a_t)$ and b_t/a_t are plotted on a base of $\rho\Omega^2/(8Pa_t^2)$. The discharge becomes zero when $\rho\Omega^2/(8Pa_t^2) = \frac{1}{4}$, i.e. when $\frac{1}{2}\Omega^2/a_t^2 = P/\rho$, and it vanishes because the cross-section of the stream disappears and the whole initial pressure is turned into swirl, leaving nothing to produce streaming velocity. At the other extreme when $\Omega \to 0$, it is found from (2.8) that $b_t^2/a_t^2 = (\frac{1}{2}\Omega/a_t)/(P/\rho)$, and from (2.9) Q becomes $\pi a_t^2(2P/\rho)^{\frac{1}{2}}$ as it should. Fig. 1 also shows that the ratio b_t/a_t, if small, is very sensitive to slight alterations in Ω and P.

For insertion into the boundary-layer equations we shall want the values along the nozzle of W, dp_a/dz, and dW/dz, p_a being the pressure at the wall. When Q has been determined, W and b may be removed from (2.2), (2.3), and (2.5), leaving an expression for p_a in the form of a cubic equation; and after this is solved, W is calculated from (2.2). An easier procedure, however, is to eliminate p_a and b from these three equations, for the resulting cubic equation for W, viz.

$$W^3 - \frac{Q}{\pi a^2}W^2 + \left(\frac{\Omega^2}{a^2} - 2\frac{P}{\rho}\right)W + 2\frac{P}{\rho}\frac{Q}{\pi a^2} = 0, \tag{2.10}$$

has simpler coefficients. One of the roots of (2.10) is negative and therefore irrelevant. The other two are positive, the smaller being the velocity upstream from the throat and the larger being the velocity below the

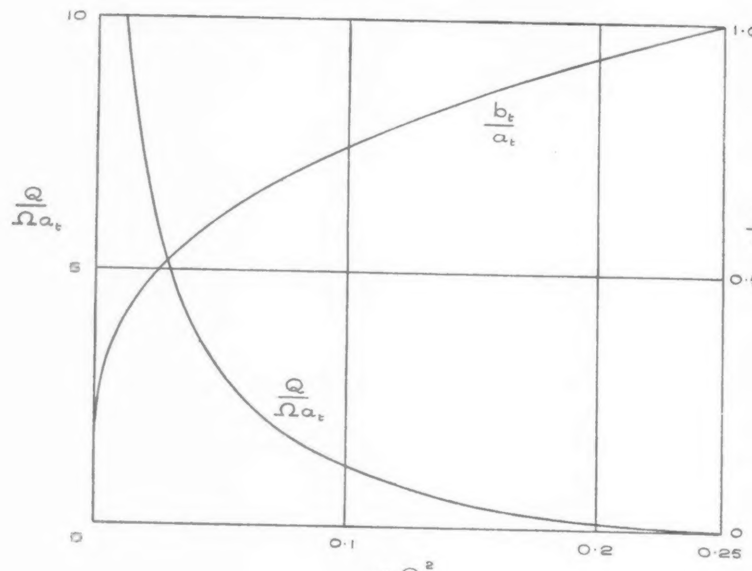

FIG. 1. Discharge and core throat radius in frictionless flow.

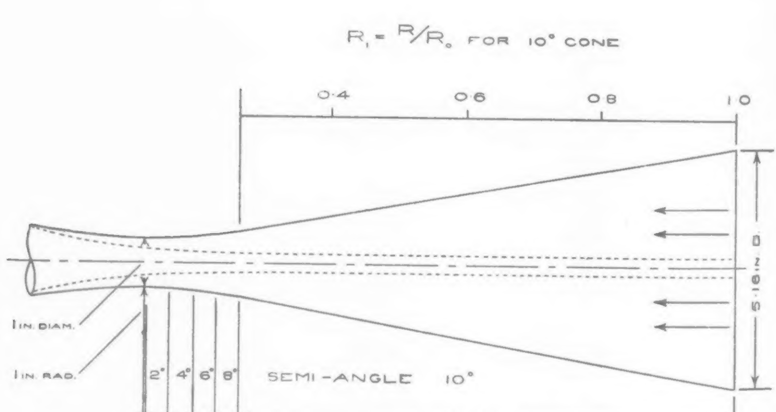

FIG. 2. Shapes of nozzle and core.

throat at the section having the same area; at the throat these two roots are equal. The velocity gradient is obtained by the differentiation of (2.10) which yields

$$\frac{dW}{dz} = \frac{\dfrac{2}{a}\dfrac{da}{dz}\left(-\dfrac{Q}{\pi a^2}W^2+\dfrac{\Omega^2}{a^2}W+2\dfrac{P}{\rho}\dfrac{Q}{\pi a^2}\right)}{3W^2-\dfrac{2Q}{\pi a^2}W+\dfrac{\Omega^2}{a^2}-2\dfrac{P}{\rho}}. \tag{2.11}$$

At the throat this expression assumes a 0/0 form, because there da/dz and the denominator are zero. On applying the usual method for dealing with this case, we find with the aid of (2.7) and (2.9) that

$$\frac{dW}{dz} = \frac{\Omega}{b_t}\left\{\frac{d^2a/dz^2}{a_t(2+b_t^2/a_t^2)}\right\}^{\frac{1}{2}}. \tag{2.12}$$

Thus the radius of curvature of the wall, which is the reciprocal of d^2a/dz^2, must be known at the throat in order that the velocity gradient may be determined. Finally the pressure gradients are calculated by differentiating (2.2).

The theory explained above is evidently subject to certain limitations. Firstly, when the angle of the cone is large the assumption of negligible radial velocities is violated. Secondly, it is a matter of common observation that in actual nozzles the discharge at high swirls does not cease; under these conditions the whole of the discharge must pass through the boundary layer. Measurements made to elucidate this point were shown by Binnie and Hookings (**3**) in their Table 1, from which it appears that the simple theory does not break down if the swirl be moderate.

As a numerical example we have considered the nozzle indicated in Fig. 2. The convergent part consists principally of a cone of inlet radius 2·59 in. and semi-angle 10°, extending for an axial length of 10·87 in. To produce parallel streaming motion at the throat, the nozzle wall is continued by a succession of conical portions having semi-angles 8°, 6°, 4°, and 2°, each of axial length $\frac{1}{2}$ in. To form the throat itself, a circular arc is placed tangentially to the 2° cone; equation (2.12) requires that the radius of this arc should be specified, and it has been taken as 1 in. The throat radius a_t is $\frac{1}{2}$ in., and the nozzle is symmetrical about the throat. The nozzle operates with water under a head of 30 ft., so that

$$P/\rho = 30\times 32{\cdot}2 \text{ ft.}^2/\text{sec.}^2,$$

and the irrotational constant Ω is 0·7 ft.²/sec. Hence

$$\rho\Omega^2/(8Pa_t^2) = 0{\cdot}0365,$$

and it is seen from Fig. 1 that the motion is far removed from the range near $\rho\Omega^2/(8Pa_t^2) = \frac{1}{4}$, where the theory is known to be unsatisfactory.

The calculations have been pursued for a short distance downstream from the throat, and the values of W, dW/dz, and dp_a/dz, which are required

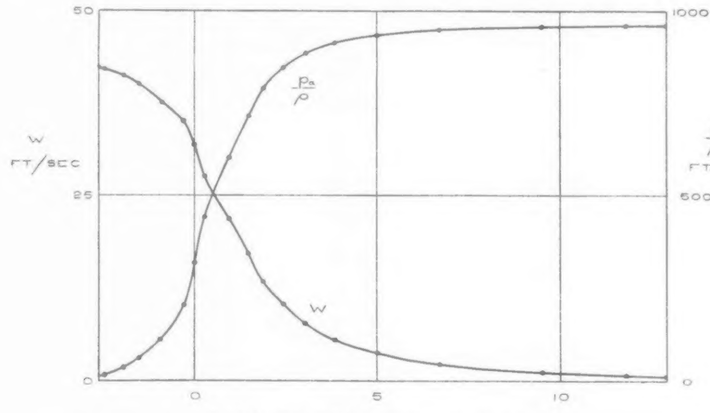

FIG. 3. Distribution of streaming velocity and wall pressure in the nozzle.

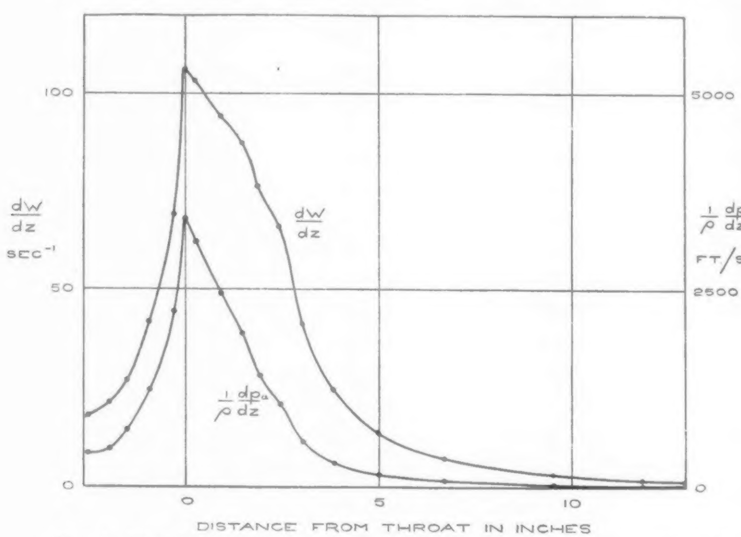

FIG. 4. Distribution of streaming velocity and wall pressure gradients in the nozzle.

for the solution of the boundary-layer equations, are displayed in Figs. 3 and 4; in the former the values of p_a are also inserted, and the shape of the air core has been added in broken lines to Fig. 2, which shows that in

the approach to the throat the core diameter is almost constant. Slight irregularities are visible, due to the absence of smooth transitions between the various conical portions of the wall; but they are not important, since the theory is approximate and, moreover, the boundary-layer equations can only be solved by a step-by-step method. The computations were found to involve small differences between large quantities, consequently the use of a calculating machine was necessary to secure sufficient accuracy. Near the throat the results are evidently sensitive to the form of the nozzle there. In a nozzle terminating abruptly at the geometrical throat, we must expect the flow to be modified, for the theory set out in this section does not contemplate a rapid reduction to zero of the wall pressure in this region.

3. Effects of surface-tension

We must now consider whether the forces due to surface-tension have appreciable effects. In section 2 the pressure over the surface of the core was taken as zero, but when surface-tension (denoted by γ) is taken into account, this pressure is

$$p_b = -\gamma\left(\frac{1}{R_1}+\frac{1}{R_2}\right), \tag{3.1}$$

where R_1 and R_2 are the principal radii of curvature. Since the free surface is a surface of revolution, R_1 is the intercept on the normal between the surface and the axis, and R_2 is the radius of the generating curve. We will confine our attention to the neighbourhood of the throat, so that R_1 may be taken as equal to b and R_2 as very large. Accordingly (3.1) reduces to

$$p_b = -\gamma/b, \tag{3.2}$$

and (2.3) becomes

$$\frac{p}{\rho} = \frac{\Omega^2}{2}\left(\frac{1}{b^2}-\frac{1}{r^2}\right)-\frac{\gamma}{\rho b}. \tag{3.3}$$

Bernoulli's equation (2.2) is unchanged, and when used with (3.3) it yields

$$W^2 = \frac{2P}{\rho}-\frac{\Omega^2}{b^2}\left(1-\frac{2b\gamma}{\rho\Omega^2}\right), \tag{3.4}$$

showing that the streaming velocity is again uniform over the annular cross-section. Employing the same procedure as before, we differentiate (2.5) and (3.4) with respect to z, and, after eliminating db/dz, we find that

$$\frac{\Omega^2 a}{b^4}\left(1-\frac{b\gamma}{\rho\Omega^2}\right)\frac{da}{dz} = \frac{W^2-W_2^2}{W}\frac{dW}{dz}, \tag{3.5}$$

where

$$W_2 = \left\{\frac{\Omega^2(a^2-b^2)}{2b^4}\left(1-\frac{b\gamma}{\rho\Omega^2}\right)\right\}^{\frac{1}{2}}. \tag{3.6}$$

The flow under consideration is seen to be impossible if $b\gamma \nless \rho\Omega^2$; and, subject to this restriction, it follows from (3.5) that at the throat $W = W_2$.

It may readily be proved that W_2 is equal to the velocity V' of a 'long' annular wave moving axially on the surface of the spinning liquid. Adapting the method employed by Binnie and Hookings, we take q to be the additional velocity due to the wave motion at points where the infinitely small surface elevation is η. The equation of continuity is then

$$\{\pi(a^2-b^2)+2\pi b\eta\}(V'+q) = \pi(a^2-b^2)V'. \tag{3.7}$$

With surface-tension operative, the increase of pressure at any radius due to the wave is

$$\frac{\Omega^2}{2}\left(\frac{1}{(b-\eta)^2}-\frac{1}{b^2}\right)-\frac{\gamma}{\rho}\left(\frac{1}{b-\eta}-\frac{1}{b}\right), \tag{3.8}$$

which reduces to

$$\frac{\Omega^2\eta}{b^3}\left(1-\frac{b\gamma}{\rho\Omega^2}\right) \tag{3.9}$$

when η is small. Hence, δp being the excess of pressure due to the wave motion,

$$\frac{\delta p}{\rho}+\frac{\Omega^2\eta}{b^3}\left(1-\frac{b\gamma}{\rho\Omega^2}\right)+\tfrac{1}{2}(V'+q)^2 = \tfrac{1}{2}V'^2. \tag{3.10}$$

On substitution from (3.7), (3.10) becomes

$$\frac{\delta p}{\rho}+\frac{\Omega^2\eta}{b^3}\left(1-\frac{b\gamma}{\rho\Omega^2}\right) = \frac{2b\eta V'^2}{a^2-b^2}, \tag{3.11}$$

and the surface condition $\delta p = 0$ is satisfied when V' has the value assigned to W_2 in (3.6). It therefore appears that the general conclusions of critical flow theory are not vitiated by the action of surface-tension.

We next attempt to determine b_t by equating the throat values of (3.4) and (3.6), but in place of a quadratic equation for b_t^2 we now arrive at the biquadratic equation

$$\frac{1}{2M}b_t^4+3Kb_t^3-b_t^2+Ka_t^2b_t-a_t^2 = 0, \tag{3.12}$$

where

$$M = \frac{\rho\Omega^2}{8P} \quad\text{and}\quad K = \frac{\gamma}{\rho\Omega^2}. \tag{3.13}$$

This may be solved by trial, a general solution not being obtainable without serious complications. Some immediate progress, however, can be made if K is assumed small, so that the solution does not differ widely from (2.8). When K is zero, (3.12) reduces to the form previously used, of which the root, say $b_t = \beta$, is given by (2.8) and shown in Fig. 1. We therefore suppose that

$$b_t = \beta(1-\epsilon), \tag{3.14}$$

where ϵ is small, and, on inserting this into (3.12) we obtain approximately

$$\epsilon = \frac{3\beta^2+a_t^2}{(2\beta/K)\{(\beta^2/M)-1\}}. \tag{3.15}$$

For water, the ratio γ/ρ (which may be termed the kinematic surface-tension) is about 0·00261 ft.^{3}/sec.^{2} In the numerical case under consideration it is found from (2.8) that $b_t = \beta = 0{\cdot}02317$ ft. The corresponding value of $\beta\gamma/(\rho\Omega^2)$ is only $1{\cdot}24\times 10^{-4}$, and we see from (3.4) and (3.6) that the effects of surface-tension are negligible. This conclusion is confirmed by (3.15) which yields $\epsilon = 0{\cdot}52\times 10^{-4}$.

4. The motion in the boundary layer

It is now necessary to employ spherical polar coordinates. The notation used is set out in Fig. 5, which indicates how the component velocities

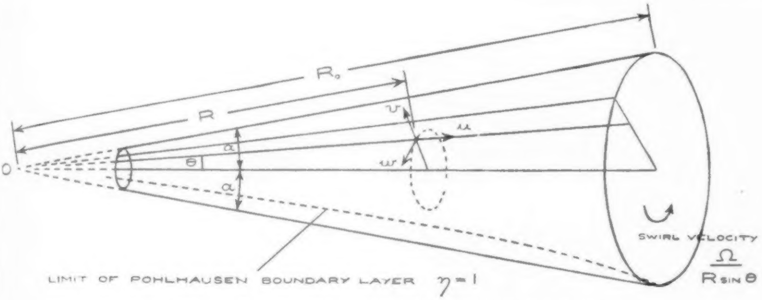

FIG. 5. System of spherical polar coordinates.

u, v, w are related to the axes. For flow symmetrical about the axis $\theta = 0$, the complete equations of steady motion for a viscous liquid are proved by Goldstein (**5**) to be

$$u\frac{\partial u}{\partial R}+\frac{v}{R}\frac{\partial u}{\partial \theta}-\frac{w^2}{R}-\frac{v^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial R}+\nu\left(\nabla^2 u-\frac{2u}{R^2}-\frac{2}{R^2}\frac{\partial v}{\partial \theta}-\frac{2v\cot\theta}{R^2}\right), \tag{4.1}$$

$$u\frac{\partial w}{\partial R}+\frac{v}{R}\frac{\partial w}{\partial \theta}+\frac{wu}{R}+\frac{vw\cot\theta}{R} = \nu\left(\nabla^2 w-\frac{w}{R^2\sin^2\theta}\right), \tag{4.2}$$

where p is the pressure, ρ and ν are the density and kinematic viscosity, and

$$\nabla^2 \equiv \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\right)+\frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right). \tag{4.3}$$

The complete equation of continuity is

$$\frac{\partial u}{\partial R}+\frac{2u}{R}+\frac{1}{R}\frac{\partial v}{\partial \theta}+\frac{v}{R}\cot\theta = 0. \tag{4.4}$$

Many of the terms in these equations are small, and, as shown by Taylor,

the forms of (4.1), (4.2), and (4.4) which may be used in an approximate solution are

$$u\frac{\partial u}{\partial R}+\frac{v}{R}\frac{\partial u}{\partial \theta}-\frac{w^2}{R}+\frac{1}{\rho}\frac{\partial p}{\partial R}=\frac{\nu}{R^2}\frac{\partial^2 u}{\partial \theta^2}, \tag{4.5}$$

$$u\frac{\partial w}{\partial R}+\frac{v}{R}\frac{\partial w}{\partial \theta}+\frac{wu}{R}=\frac{\nu}{R^2}\frac{\partial^2 w}{\partial \theta^2}, \tag{4.6}$$

$$\frac{\partial u}{\partial R}+2\frac{u}{R}+\frac{1}{R}\frac{\partial v}{\partial \theta}=0. \tag{4.7}$$

We assume that the boundary layer is thin and that the pressure through its thickness is the same as in the adjoining main stream. It was shown in section 2 how an expression for this pressure could be derived, but it is too complicated to be of service here, and for the time being we shall take the pressure to be given by

$$\frac{1}{\rho}\frac{\partial p_a}{\partial R}=F(R), \tag{4.8}$$

where $F(R)$ is the function of R calculated from (2.10), (2.11), and (2.2). For the same reason we suppose that u in the main stream close to the boundary layer is of the form

$$u=G(R). \tag{4.9}$$

The boundary layer is taken to have a definite thickness δ which is a function of R, and the momentum integrals are obtained by integrating the equations of motion (4.5) and (4.6) between the limits $\theta=\alpha$ at the wall and $\theta=\alpha-\delta/R$ at the edge of the main stream. In this way (4.5) and (4.8) yield

$$\int\limits_{\alpha-\delta/R}^{\alpha} u\frac{\partial u}{\partial R}\,d\theta+\int\limits_{\alpha-\delta/R}^{\alpha}\frac{v}{R}\frac{\partial u}{\partial \theta}\,d\theta+\int\limits_{\alpha-\delta/R}^{\alpha}\left(F-\frac{w^2}{R}\right)d\theta=\frac{\nu}{R^2}\left[\frac{\partial u}{\partial \theta}\right]_{\theta=\alpha}, \tag{4.10}$$

$\partial u/\partial\theta$ at $\theta=\alpha-\delta/R$ being supposed zero. From this equation v can be removed with the aid of (4.7), for

$$\int\limits_{\alpha-\delta/R}^{\alpha}\frac{v}{R}\frac{\partial u}{\partial \theta}\,d\theta=\int\limits_{\alpha-\delta/R}^{\alpha}\frac{1}{R}\left\{\frac{\partial(uv)}{\partial \theta}-u\frac{\partial v}{\partial \theta}\right\}d\theta$$

$$=\frac{1}{R}[uv]^{\alpha}_{\alpha-\delta/R}+\int\limits_{\alpha-\delta/R}^{\alpha}\left(u\frac{\partial u}{\partial R}+\frac{2u^2}{R}\right)d\theta. \tag{4.11}$$

To evaluate $v_{\alpha-\delta/R}$ we integrate (4.7) which yields

$$v_{\alpha-\delta/R}=\int\limits_{\alpha-\delta/R}^{\alpha}\left(R\frac{\partial u}{\partial R}+2u\right)d\theta \tag{4.12}$$

because $v_\alpha = 0$, and thus we find on using (4.9) that

$$[uv]^{\alpha}_{\alpha-\delta/R} = -G\int\limits_{\alpha-\delta/R}^{\alpha}\left(R\frac{\partial u}{\partial R}+2u\right)d\theta. \tag{4.13}$$

When (4.13) and (4.11) are inserted in (4.10), the result is the longitudinal momentum integral

$$2\int\limits_{\alpha-\delta/R}^{\alpha} u\frac{\partial u}{\partial R}\,d\theta+2\int\limits_{\alpha-\delta/R}^{\alpha}\frac{u^2}{R}\,d\theta-\frac{G}{R}\int\limits_{\alpha-\delta/R}^{\alpha}\left(R\frac{\partial u}{\partial R}+2u\right)d\theta+$$
$$+\int\limits_{\alpha-\delta/R}^{\alpha}\left(F-\frac{w^2}{R}\right)d\theta = \frac{\nu}{R^2}\left[\frac{\partial u}{\partial\theta}\right]_{\theta=\alpha}. \tag{4.14}$$

The same method can be applied to (4.6) which, with the assistance of a relation analogous to (4.11), yields

$$\int\limits_{\alpha-\delta/R}^{\alpha} u\frac{\partial w}{\partial R}\,d\theta+\frac{1}{R}[vw]^{\alpha}_{\alpha-\delta/R}+\int\limits_{\alpha-\delta/R}^{\alpha}\left(w\frac{\partial u}{\partial R}+\frac{2uw}{R}\right)d\theta+$$
$$+\int\limits_{\alpha-\delta/R}^{\alpha}\frac{wu}{R}\,d\theta = \frac{\nu}{R^2}\left[\frac{\partial w}{\partial\theta}\right]_{\theta=\alpha}, \tag{4.15}$$

for $\partial w/\partial\theta$ at $\theta = \alpha-\delta/R$ is taken to be zero. At this limit $w = \Omega/(R\sin\alpha)$ and v is given by (4.12), hence (4.15) becomes

$$\int\limits_{\alpha-\delta/R}^{\alpha}\frac{\partial(uw)}{\partial R}\,d\theta-\frac{\Omega}{R^2\sin\alpha}\int\limits_{\alpha-\delta/R}^{\alpha}\left(R\frac{\partial u}{\partial R}+2u\right)d\theta+3\int\limits_{\alpha-\delta/R}^{\alpha}\frac{uw}{R}\,d\theta = \frac{\nu}{R^2}\left[\frac{\partial w}{\partial\theta}\right]_{\theta=\alpha}, \tag{4.16}$$

which is the transverse momentum integral.

We now use η instead of θ as a variable, η being defined by

$$\eta = \frac{R}{\delta}(\alpha-\theta) \tag{4.17}$$

so that it varies from 0 to 1 as we pass from the wall to the edge of the boundary layer. At this stage it is necessary to adopt the Pohlhausen method of approximation, in which simple arbitrary assumptions, consistent with the boundary conditions, are made for the velocity distribution in the boundary layer. Accordingly we suppose that

$$u = G(R)\phi(\eta) \tag{4.18}$$

and

$$w = \frac{\Omega}{R\sin\alpha}\phi(\eta), \tag{4.19}$$

where

$$\phi(\eta) = 2\eta-\eta^2. \tag{4.20}$$

It will be seen that (4.18) and (4.19) satisfy the boundary conditions $u = w = 0$ at $\eta = 0$; $u = G(R)$, $w = \Omega/(R\sin\alpha)$, $\partial u/\partial\eta = \partial w/\partial\eta = 0$ at $\eta = 1$.

Before the integrals (4.14) and (4.16) can be evaluated, $\partial u/\partial R$ and $\partial(uw)/\partial R$, which are differentials at constant θ, must be expressed in terms of ϕ, δ, and η. We have from (4.18) that

$$\left(\frac{\partial u}{\partial R}\right)_\theta = G\frac{d\phi}{d\eta}\left(\frac{\partial\eta}{\partial R}\right)_\theta + \phi\frac{dG}{dR}, \tag{4.21}$$

but from (4.17)

$$\left(\frac{\partial\eta}{\partial R}\right)_\theta = \frac{\alpha-\theta}{\delta} - \frac{R}{\delta^2}(\alpha-\theta)\frac{d\delta}{dR} = \frac{\eta}{R}\left(1-\frac{R}{\delta}\frac{d\delta}{dR}\right), \tag{4.22}$$

hence

$$\left(\frac{\partial u}{\partial R}\right)_\theta = G\frac{d\phi}{d\eta}\frac{\eta}{R}\left(1-\frac{R}{\delta}\frac{d\delta}{dR}\right) + \phi\frac{dG}{dR}. \tag{4.23}$$

Similarly, from (4.18) and (4.19)

$$\frac{\sin\alpha}{\Omega}\left(\frac{\partial(uw)}{\partial R}\right)_\theta = \frac{2G\phi}{R}\frac{d\phi}{d\eta}\frac{\eta}{R}\left(1-\frac{R}{\delta}\frac{d\delta}{dR}\right) + \phi^2\left(\frac{1}{R}\frac{dG}{dR} - \frac{G}{R^2}\right). \tag{4.24}$$

We now change the variable in (4.14) and (4.16) from $d\theta$ to $-(\delta/R)\,d\eta$ and utilize (4.23) and (4.24). A simplification is effected by means of the expressions

$$\int_0^1 \eta\phi\frac{d\phi}{d\eta}\,d\eta = \tfrac{1}{2} - \int_0^1 \frac{\phi^2}{2}\,d\eta \tag{4.25}$$

and

$$\int_0^1 \eta\frac{d\phi}{d\eta}\,d\eta = 1 - \int_0^1 \phi\,d\eta, \tag{4.26}$$

which are obtained on integrating by parts and making use of the boundary conditions. The momentum integrals then reduce to

$$\left(\frac{G}{R}+\frac{dG}{dR}+\frac{G}{2\delta^2}\frac{d\delta^2}{dR}\right)\left(\int_0^1 \phi\,d\eta - \int_0^1 \phi^2\,d\eta\right) - \left(\frac{dG}{dR} - \frac{\Omega^2}{GR^3\sin^2\alpha}\right)\int_0^1 \phi^2\,d\eta - \frac{F}{G}$$
$$= \frac{\nu}{\delta^2}\left[\frac{d\phi}{d\eta}\right]_{\eta=0}, \tag{4.27}$$

$$\left(\frac{G}{R}+\frac{dG}{dR}+\frac{G}{2\delta^2}\frac{d\delta^2}{dR}\right)\left(\int_0^1 \phi\,d\eta - \int_0^1 \phi^2\,d\eta\right) = \frac{\nu}{\delta^2}\left[\frac{d\phi}{d\eta}\right]_{\eta=0}. \tag{4.28}$$

These may be put in non-dimensional form by substituting

$$R_1 = \frac{R}{R_0}, \qquad \delta_1 = \frac{\delta}{R_0}\left(\frac{\Omega}{\nu\sin\alpha}\right)^{\frac{1}{2}}, \tag{4.29}$$

and replacing (4.8) and (4.9) by

$$\frac{1}{\rho}\frac{\partial p_a}{\partial R} = F(R) = \frac{\Omega^2}{R_0^3 \sin^2\alpha} f(R_1) \tag{4.30}$$

and

$$u = G(R) = \frac{\Omega}{R_0 \sin\alpha} g(R_1), \tag{4.31}$$

where R_0 is the distance along the generator from the apex to the base of the cone. Then (4.27) and (4.28) become

$$\left(\frac{g}{R_1}+\frac{dg}{dR_1}+\frac{g}{2\delta_1^2}\frac{d\delta_1^2}{dR_1}\right)\left(\int_0^1 \phi\, d\eta - \int_0^1 \phi^2\, d\eta\right) - \left(\frac{dg}{dR_1}-\frac{1}{gR_1^3}\right)\int_0^1 \phi^2\, d\eta - \frac{f}{g} = \frac{1}{\delta_1^2}\left[\frac{d\phi}{d\eta}\right]_{\eta=0}, \tag{4.32}$$

$$\left(\frac{g}{R_1}+\frac{dg}{dR_1}+\frac{g}{2\delta_1^2}\frac{d\delta_1^2}{dR_1}\right)\left(\int_0^1 \phi\, d\eta - \int_0^1 \phi^2\, d\eta\right) = \frac{1}{\delta_1^2}\left[\frac{d\phi}{d\eta}\right]_{\eta=0}. \tag{4.33}$$

Apart from boundary conditions we have not yet employed the assumed expression for ϕ indicated in (4.20). With this particular form

$$\int_0^1 \phi^2\, d\eta = 0{\cdot}5\dot{3}, \qquad \int_0^1 \phi\, d\eta = 0{\cdot}\dot{6}, \qquad \left[\frac{d\phi}{d\eta}\right]_{\eta=0} = 2; \tag{4.34}$$

in that case (4.32) and (4.33) reduce to

$$\frac{g}{\delta_1^2}\frac{d\delta_1^2}{dR_1} - 6\frac{dg}{dR_1} + 2\frac{g}{R_1} + \frac{8}{gR_1^3} - \frac{15f}{g} = \frac{30}{\delta_1^2}, \tag{4.35}$$

$$\frac{g}{2\delta_1^2}\frac{d\delta_1^2}{dR_1} + \frac{dg}{dR_1} + \frac{g}{R_1} = \frac{15}{\delta_1^2}. \tag{4.36}$$

If we replace (4.20) by the more elaborate expression

$$\phi = 2\eta - 2\eta^3 + \eta^4, \tag{4.37}$$

which also satisfies the boundary conditions, we find that

$$\int_0^1 \phi^2\, d\eta = 0{\cdot}5825, \qquad \int_0^1 \phi\, d\eta = 0{\cdot}7, \qquad \left[\frac{d\phi}{d\eta}\right]_{\eta=0} = 2, \tag{4.38}$$

and that (4.35) and (4.36) are changed to

$$\frac{g}{\delta_1^2}\frac{d\delta_1^2}{dR_1} - 7{\cdot}915\frac{dg}{dR_1} + 2\frac{g}{R_1} + \frac{9{\cdot}915}{gR_1^3} - 17{\cdot}02\frac{f}{g} = \frac{34{\cdot}04}{\delta_1^2}, \tag{4.39}$$

$$\frac{g}{2\delta_1^2}\frac{d\delta_1^2}{dR_1} + \frac{dg}{dR_1} + \frac{g}{R_1} = \frac{17{\cdot}02}{\delta_1^2}. \tag{4.40}$$

It will be seen that the alterations in the coefficients are not large. Some idea of the magnitude of the approximations can be obtained from the subtraction of (4.27) from (4.28) which yields

$$G\frac{dG}{dR}-\frac{\Omega^2}{R^3\sin^2\alpha}+F\Big/\int\limits_0^1\phi^2\,d\eta=0. \tag{4.41}$$

The coefficients of F in the two cases considered are 1·875 and 1·717. For comparison an expression, which is exact as far as this section of the work is concerned, can be derived by differentiating (2.2) after putting $r = a$. It then follows that

$$G\frac{dG}{dR}-\frac{\Omega^2}{R^3\sin^2\alpha}+F=0, \tag{4.42}$$

indicating a discrepancy in the final term.

Evidently the solution obtained above is not valid when the streaming velocity is zero. In those circumstances (4.18) shows that u would also be zero, whereas Taylor's theory yields a definite expression for this velocity. However, at the other extreme, where $\Omega \to 0$, a check on the work can be obtained by comparing the results with those due to Mangler (**6**) and in the next section the methods used above will be applied to this case.

5. Boundary-layer theory when the swirl is zero

We are now concerned with simple flow without swirl through the nozzle. As a first step it is necessary to determine the conditions in the main stream, using the approximations and notation of section 2. Taking the throat pressure to be zero, we have

$$Q=\pi a_t^2\left(\frac{2P}{\rho}\right)^{\frac{1}{2}}, \tag{5.1}$$

and it follows that
$$W=\frac{a_t^2}{a^2}\left(\frac{2P}{\rho}\right)^{\frac{1}{2}}, \tag{5.2}$$

whence from (2.2),
$$\frac{p_a}{\rho}=\frac{P}{\rho}\left(1-\frac{a_t^4}{a^4}\right). \tag{5.3}$$

Accordingly (4.8) and (4.9) are replaced by

$$\frac{1}{\rho}\frac{\partial p_a}{\partial R}=F(R)=\frac{4a_t^4}{R^5\sin^4\alpha}\frac{P}{\rho} \tag{5.4}$$

and
$$u=G(R)=-\frac{a_t^2}{R^2\sin^2\alpha}\left(\frac{2P}{\rho}\right)^{\frac{1}{2}}. \tag{5.5}$$

Equation (4.6) disappears; and as (4.5) leads to (4.27), in the latter we substitute (5.4) and (5.5) and put $\Omega = 0$. The result is

$$\left(1-\frac{R}{2\delta^2}\frac{d\delta^2}{dR}\right)\left(\int_0^1 \phi\, d\eta-\int_0^1 \phi^2\, d\eta\right)-2\int_0^1 \phi^2\, d\eta+2 = \frac{\nu}{\delta^2}\frac{R^3 \sin^2\alpha}{a_t^2(2P/\rho)^{\frac{1}{2}}}\left[\frac{d\phi}{d\eta}\right]_{\eta=0}. \tag{5.6}$$

To get this into non-dimensional form we write

$$R_1 = \frac{R}{R_0}, \qquad \delta_2 = \delta\left\{\frac{a_t^2(2P/\rho)^{\frac{1}{2}}}{\nu R_0^3 \sin^2\alpha}\right\}^{\frac{1}{2}}; \tag{5.7}$$

then (5.6) reduces to

$$\left(1-\frac{R_1}{2\delta_2^2}\frac{d\delta_2^2}{dR_1}\right)\left(\int_0^1 \phi\, d\eta-\int_0^1 \phi^2\, d\eta\right)-2\int_0^1 \phi^2\, d\eta+2 = \frac{R_1^3}{\delta_2^2}\left[\frac{d\phi}{d\eta}\right]_{\eta=0}. \tag{5.8}$$

With the numerical values given in (4.34) this equation for δ_2^2 becomes

$$\frac{d\delta_2^2}{dR_1}-16\frac{\delta_2^2}{R_1} = -30R_1^2, \tag{5.9}$$

the solution of which, subject to the condition that $\delta_2 = 0$ at $R_1 = 1$, is

$$\delta_2^2 = \tfrac{30}{13}R_1^3(1-R_1^{13}). \tag{5.10}$$

The term R_1^{13} is appreciable only in the initial stage, and over the rest of the cone the momentum thickness is

$$\vartheta = (\tfrac{30}{13})^{\frac{1}{2}}R_1^{\frac{3}{2}}\int_0^1 (\phi-\phi^2)\, d\eta = 0{\cdot}203R_1^{\frac{3}{2}}. \tag{5.11}$$

Mr. E. J. Watson has pointed out to us that, for comparison with (5.10) and (5.11), use may be made of Mangler's work (**6**) on *exact* solutions of axially symmetrical boundary-layer equations. As a special case of his general theory, Mangler considers radial flow over an unlimited flat plate towards a central hole, and since the velocity in the main stream varies inversely as the square of the distance from the origin, his analysis applies also to a cone of small angle. His boundary conditions are different from ours, for both at the origin and also at an infinite distance from the wall, he takes the whole of the boundary layer to have the same tangential velocity as the main stream. It can then be shown that the momentum thickness is $0{\cdot}291R_1^{\frac{3}{2}}$. The agreement is closer when displacement thicknesses are compared; (5.10) then leads to $0{\cdot}506R_1^{\frac{3}{2}}$, while Mangler's result is $0{\cdot}587R_1^{\frac{3}{2}}$.

6. Numerical example of boundary-layer calculation

For the purpose of step-by-step calculation (4.35) may be written in the form

$$\frac{d\delta_1^2}{dR_1}+A\,\delta_1^2=\frac{30}{g}, \tag{6.1}$$

where

$$A=\frac{1}{g}\left(-6\frac{dg}{dR_1}+2\frac{g}{R_1}+\frac{8}{gR_1^3}-15\frac{f}{g}\right) \tag{6.2}$$

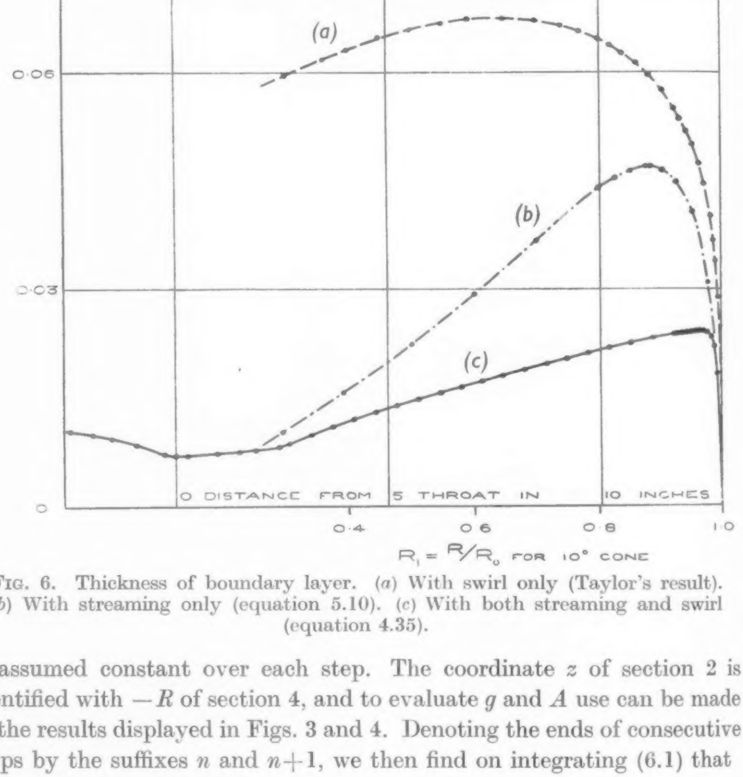

Fig. 6. Thickness of boundary layer. (*a*) With swirl only (Taylor's result). (*b*) With streaming only (equation 5.10). (*c*) With both streaming and swirl (equation 4.35).

is assumed constant over each step. The coordinate z of section 2 is identified with $-R$ of section 4, and to evaluate g and A use can be made of the results displayed in Figs. 3 and 4. Denoting the ends of consecutive steps by the suffixes n and $n+1$, we then find on integrating (6.1) that

$$\frac{(\delta_1^2)_{n+1}-30/(g_{n+1}A)}{(\delta_1^2)_n-30/(g_n A)}=\exp[-A\{(R_1)_{n+1}-(R_1)_n\}]. \tag{6.3}$$

The process was carried out starting at the point $\delta_1=0$, $R_1=1$, adjustment of the origin of R_1 being required at the sections where the cone angle alters. The outcome is shown in dimensional units by curve (*c*) in Fig. 6, in which the ends of the steps are indicated by dots. At the throat the boundary layer has a thickness of 0·00695 in., and it occupies

a fraction 1/24·85 of the cross-section of the stream. For purposes of comparison two other curves, extending over the 10° portion of the nozzle, have been added to Fig. 6. The upper is Taylor's result for zero streaming velocity and the same swirl $\Omega = 0{\cdot}7$ ft.^{2}/sec. as in the motion now under consideration, and the lower is calculated for zero swirl from (5.10), the reservoir pressure being chosen to produce the same streaming velocity at the throat as in our case.

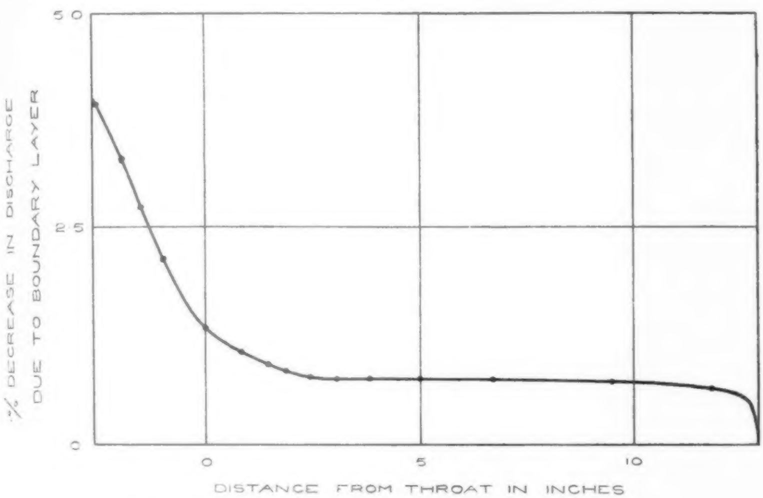

Fig. 7. Reduction of discharge due to boundary layer.

In most boundary layer calculations which have hitherto been carried out, the field occupied by the fluid has been partially infinite in extent. Here, however, the field is enclosed, and the condition exists that the total discharges over all cross-sections of the nozzle must be identical. It is therefore desirable that the reduction in discharge due to the retarded layer should be estimated and compared with the frictionless discharge Q. This reduction is given by

$$W 2\pi a\,\delta \int_0^1 (1-\phi)\,d\eta = \tfrac{2}{3} W \pi a \delta, \tag{6.4}$$

and is plotted in Fig. 7 as a percentage of Q. It is seen to be very small and almost constant in most of the convergent part of the nozzle; at the throat it is still insignificant, and it could be compensated by a reduction of only 0·00416 in. in the core radius $b_t = 0{\cdot}278$ in. Thus the flow in the main stream is almost unaffected by the boundary layer.

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NUMERICAL EVALUATION OF INTEGRALS OF THE FORM $I = \int\limits_{x_1}^{x_2} f(x)e^{i\phi(x)}\,dx$ AND THE TABULATION OF THE FUNCTION $\mathrm{Gi}(z) = (1/\pi)\int\limits_0^\infty \sin(uz+\frac{1}{3}u^3)\,du$

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SUMMARY

This type of integral often occurs in the calculation of the wave form due to a source in a dispersive medium. The principle of a stationary phase is used to evaluate it in terms of the Airy Integral, of which tables are published, and of

$$\mathrm{Gi}(z) = \frac{1}{\pi}\int\limits_0^\infty \sin(uz+\tfrac{1}{3}u^3)\,du.$$

A table is given of this function for $z > 0$ and of a closely associated function for $z < 0$. In a separate note by Dr. J. C. P. Miller it is shown that these, together with the published tables of the Airy Integral, provide the solution of the differential equation

$$\frac{d^2y}{dz^2} - zy = P(z),$$

where $P(z)$ is a polynomial or a power series.

1. Introduction

INTEGRALS of the form

$$I = \int\limits_{x_1}^{x_2} f(x)e^{i\phi(x)}\,dx$$

occur in calculating the wave form in a dispersive medium due to a source which can be represented by a Fourier integral. They may often be evaluated by making use of the well-known principle of stationary phase. In this treatment such a method is assumed to be applicable. It depends upon finding x_0 to satisfy

$$\phi_0' \equiv \phi'(x_0) = 0, \tag{1.1}$$

where dashes denote differentiation with respect to x. If $f(x)$ is slowly varying near $x = x_0$ and $e^{i\phi(x)}$ is sufficiently oscillatory elsewhere, then only in the neighbourhood of x_0 is significant contribution made to the integral. If (1.1) has more than one root in the range of integration the contributions must be added, but we may suppose without loss of generality that there is now only one.

The standard procedure is to write

$$I = f(x_0)\int\limits_{x_1}^{x_2} \exp\{i(\phi_0+\tfrac{1}{2}(x-x_0)^2\phi_0'')\}\,dx, \tag{1.2}$$

which is expressed in terms of

$$\int_{-\infty}^{\infty} \exp(ix^2)\,dx,$$

whose value is known. Such an approximation is only valid when the succeeding terms in the Taylor expansion of $\phi(x)$, viz.

$$\tfrac{1}{6}(x-x_0)^3\phi_0'''+\dots$$

are negligible over a sufficient range of x. The criterion as given by Lamb (**1**) is that

$$|\phi_0'''| \ll \{|\phi_0''|\}^{\frac{3}{2}}. \tag{1.3}$$

The following method was devised for a special problem in which (1.3) did not hold.

2. Formula for the integral

When $\phi_0''=0$, a method using the Airy Integral given by Jeffreys (**2**) may be used. That method is now generalized to apply when it is permissible to write

$$\phi(x) = \phi_0+\tfrac{1}{2}(x-x_0)^2\phi_0''+\tfrac{1}{6}(x-x_0)^3\phi_0''' \tag{2.1}$$
$$= a+bx+cx^2+dx^3, \text{ say,}$$

a, b, c, d being thus known functions of x_0. Let

$$K(z) = \mathrm{Ai}(z)+i\,\mathrm{Gi}(z) = \frac{1}{\pi}\int_{0}^{\infty} \exp\{i(uz+\tfrac{1}{3}u^3)\}\,du, \tag{2.2}$$

then if two new numbers l and m are introduced, where

$$u = l(x-m),$$

the exponent becomes a cubic in x which may be identified, apart from a constant, with $i\phi(x)$ as expressed in (2.1). For (2.2) becomes

$$K(z) = \frac{l}{\pi}\int_{m}^{\infty} \exp\{i[-zlm-\tfrac{1}{3}l^3m^3+(zl+l^3m^2)x-l^3mx^2+\tfrac{1}{3}l^3x^3]\}\,dx, \tag{2.3}$$

and we are to assume that

$$I = f(x_0)\int_{x_1}^{x_2} \exp\{i(a+bx+cx^2+dx^3)\}\,dx;$$

wherefore if also

$$\left.\begin{aligned} b &= zl+l^3m^2 \\ c &= -l^3m \\ d &= \tfrac{1}{3}l^3 \end{aligned}\right\} \qquad \text{i.e. } \left.\begin{aligned} l &= (3d)^{\frac{1}{3}} \\ m &= -c/3d \\ z &= (3d)^{-\frac{1}{3}}(b-c^2/3d) \end{aligned}\right\} \tag{2.4}$$

and provided that either m is close to x_1 when x_0 is near x_1, or x_0 is not near to x_1 or x_2, we have

$$I = \frac{\pi}{l} f(x_0) e^{i\beta} K(z), \tag{2.5}$$

where

$$\beta = a + zlm + \tfrac{1}{3} l^3 m^3 = a - \tfrac{1}{3}c - \frac{bc}{3d} + \frac{c^3}{9d^2}. \tag{2.6}$$

Here l, β, and z are functions of x_0 that are readily calculated from (2.1), (2.4), and (2.6) if $\phi(x)$ can be differentiated. $\mathrm{Ai}(z)$ is already tabulated (3) and so I is given by (2.5) when $\mathrm{Gi}(z)$ is tabulated.

3. Calculation of Gi(z)

The problem is thus reduced to that of evaluating

$$\mathrm{Gi}(z) = \frac{1}{\pi} \int_0^\infty \sin(uz + \tfrac{1}{3}u^3)\, du, \tag{3.1}$$

which is a particular integral of

$$\frac{d^2y}{dz^2} - zy = -\frac{1}{\pi}, \tag{3.2}$$

and this equation may be integrated numerically if $\mathrm{Gi}(0)$ and $\mathrm{Gi}'(0)$ are known.

Since $\mathrm{Gi}(z)$ oscillates for $z < 0$ it is more convenient for interpolation purposes to tabulate, for $z < 0$, the function

$$\mathrm{Hi}(z) = \frac{1}{\pi} \int_0^\infty \exp(uz - \tfrac{1}{3}u^3)\, du \tag{3.3}$$

$$= \mathrm{Bi}(z) - \mathrm{Gi}(z), \tag{3.4}$$

since $\mathrm{Bi}(z)$ is already tabulated in (**3**). It is readily shown by contour integration using the method indicated by Miller (**3**) that

$$\mathrm{Gi}(0) = \tfrac{1}{2}\mathrm{Hi}(0) = \tfrac{1}{3}\mathrm{Bi}(0) = 3^{-\frac{1}{2}}\mathrm{Ai}(0) = 3^{-\frac{7}{6}}/(-\tfrac{1}{3})! \tag{3.5}$$

$$\mathrm{Gi}'(0) = \tfrac{1}{2}\mathrm{Hi}'(0) = \tfrac{1}{3}\mathrm{Bi}'(0) = -3^{-\frac{1}{2}}\mathrm{Ai}'(0) = 3^{-\frac{5}{6}}/(-\tfrac{2}{3})! \tag{3.6}$$

To twelve decimals the values are

$$\mathrm{Gi}(0) = 0{\cdot}20497\ 55424\ 78 \tag{3.7}$$

$$\mathrm{Gi}'(0) = 0{\cdot}14942\ 94524\ 49. \tag{3.8}$$

To begin the integration the general power series solution of (3.2), viz.

$$y = y_0\left(1 + \frac{1\,.\,z^3}{3!} + \frac{1\,.\,4z^6}{6!} + \ldots\right) + \\ + y_0'\left(z + \frac{2\,.\,z^4}{4!} + \frac{2\,.\,5z^7}{7!} + \ldots\right) - \\ - \frac{1}{\pi}\left(\frac{z^2}{2!} + \frac{3z^5}{5!} + \frac{3\,.\,6z^8}{8!} + \ldots\right) \tag{3.9}$$

was used to determine the values of Gi(z) for the first two steps. Now (3.9) can be written

$$\mathrm{Gi}(z) = \alpha\,\mathrm{Ai}(z)+\beta\,\mathrm{Bi}(z)-\frac{1}{\pi}\left(\frac{z^2}{2!}+\frac{3z^5}{5!}+\frac{3\,.\,6z^8}{8!}+\ldots\right),$$

where α and β are numerical constants.

Hence

$$\mathrm{Gi}(0) = \alpha\,\mathrm{Ai}(0)+\beta\,\mathrm{Bi}(0)$$

$$\mathrm{Gi}'(0) = \alpha\,\mathrm{Ai}'(0)+\beta\,\mathrm{Bi}'(0),$$

and comparing these with (3.5) and (3.6) we find $\alpha = 0$, $\beta = \frac{1}{3}$, i.e.

$$\mathrm{Gi}(z) = \tfrac{1}{3}\mathrm{Bi}(z)-\frac{1}{\pi}\left(\frac{z^2}{2!}+\frac{3z^5}{5!}+\frac{3\,.\,6z^8}{8!}+\ldots\right). \tag{3.10}$$

The integration was performed by using the formula

$$\delta^2 y = (\delta z)^2\left[1+\frac{1}{12}\delta^2-\frac{1}{240}\delta^4+\frac{31}{60480}\delta^6-\ldots\right]\frac{d^2y}{dz^2}. \tag{3.11}$$

The values of the differences on the right-hand side were obtained at each stage where necessary by performing the integration a few steps ahead as necessary with progressively fewer significant figures and thus also fewer terms in (3.11).

The functions were tabulated in the range $-10 \leqslant z \leqslant 10$, outside which the asymptotic formulae give at least 9 correct decimals. Ten decimals were used in the integration and the asymptotic formula was used at $z = 8, 9, 10$ together with the series (3.9) at $z = 5$ to eliminate rounding off errors which grow like Bi(z) in forward integration for $z > 2$. For $z < 0$ the unwanted solutions of the differential equation (4.2), below, satisfied by Hi(z), are oscillatory and do not grow unduly. A programme had been made for tabulating Ai($-z$), by integrating the differential equation which it satisfies, on the EDSAC at Cambridge; and no major modification was required to tabulate Hi($-z$). The table of Hi($-z$) was therefore made on the EDSAC using an integration interval of 0·02, every fifth value being printed. The method has been described by Wilkes in (**4**).

To ensure that the oscillatory errors were small the integration was carried to $z = -12$ and compared with the results obtained by hand for $-10 \geqslant z \geqslant -12$ starting from the asymptotic formula. Eight decimals correct were obtained throughout, and a larger table is contemplated to contain these; but for present purposes, where only modified second differences are given for interpolation, seven decimals are set down.

4. Asymptotic formulae

The asymptotic form of Gi(z) can contain no multiple of Bi(z) when $z \to \infty$ since Bi(z) increases indefinitely. Now a particular integral of (3.2) is

$$y \sim \frac{1}{\pi}\left(\frac{1}{z}+\frac{2!}{z^4}+\frac{5!}{3z^7}+\frac{8!}{3\,.\,6z^{10}}+...\right),$$

which is of a higher order of magnitude than Ai(z) whose asymptotic form is

$$\frac{1}{2\sqrt{\pi}}z^{-\frac{1}{4}}\exp(-\tfrac{2}{3}z^{\frac{3}{2}})(1-\tfrac{5}{48}z^{-\frac{3}{2}}+...).$$

Thus, in the Poincaré sense, for $z > 0$

$$\mathrm{Gi}(z) \sim \frac{1}{\pi}\left(\frac{1}{z}+\frac{2!}{z^4}+\frac{5!}{3z^7}+\frac{8!}{3\,.\,6z^{10}}+...\right), \tag{4.1}$$

and the numerical work has confirmed that no multiple of Ai(z) enters into the formula.

When $z < 0$, writing $z = -\zeta$, we have (see (**2**) or (**3**))

$$-\mathrm{Hi}(\zeta) = \mathrm{Gi}(\zeta)-\mathrm{Bi}(\zeta) = \frac{1}{\pi}\int\limits_0^\infty \exp(-u\zeta-\tfrac{1}{3}u^3)\,du.$$

Integrating by parts, the first two or three terms of the asymptotic expansion of the right-hand side can be obtained, the subsequent terms being found by substitution in the differential equation satisfied by Hi(z), namely

$$\frac{d^2y}{dz^2}-zy = \frac{1}{\pi}. \tag{4.2}$$

Thus, for $z = -\zeta$,

$$\mathrm{Hi}(\zeta) = \mathrm{Bi}(\zeta)-\mathrm{Gi}(\zeta) \sim \frac{1}{\pi}\left(\frac{1}{\zeta}-\frac{2!}{\zeta^4}+\frac{5!}{3\zeta^7}-\frac{8!}{3\,.\,6\zeta^{10}}+...\right). \tag{4.3}$$

5. Acknowledgements

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Table of Hi($-z$) *and* Gi(z)

z	Hi($-z$)	δ^2_m	Gi(z)	δ^2_m	z	Hi($-z$)	δ^2_m	Gi(z)	δ^2_m
0·0	0·40995 11	+317 69	0·20497 55	−318 60	5·0	0·06276 33	+4 48	0·06491 98	+6 28
0·1	·38159 08	279 64	·21836 23	296 70	5·1	·06157 85	4 26	·06356 46	5 86
0·2	·35603 63	246 67	·22878 60	272 69	5·2	·06043 63	4 04	·06226 81	5 47
0·3	·33295 65	218 04	·23648 52	247 41	5·3	·05933 45	3 84	·06102 64	5 14
0·4	·31206 40	193 17	·24171 13	221 64	5·4	·05827 11	3 65	·05983 61	4 80
0·5	0·29310 91	+171 45	0·24472 10	−195 88	5·5	0·05724 42	+3 45	0·05869 39	+4 51
0·6	·27587 39	152 56	·24577 08	170 75	5·6	·05625 19	3 32	·05759 68	4 22
0·7	·26016 86	135 99	·24511 13	146 62	5·7	·05529 27	3 12	·05654 20	3 98
0·8	·24582 70	121 46	·24298 33	123 75	5·8	·05436 48	3 02	·05552 70	3 75
0·9	·23270 33	108 71	·23961 49	102 51	5·9	·05346 70	2 83	·05454 95	3 53
1·0	0·22066 96	+ 97 52	0·23521 84	− 82 89	6·0	0·05259 76	+2 75	0·05360 73	+3 34
1·1	·20961 35	87 60	·22998 96	65 15	6·1	·05175 56	2 58	·05269 85	3 13
1·2	·19943 56	78 90	·22410 60	49 21	6·2	·05093 95	2 49	·05182 11	3 01
1·3	·19004 85	71 16	·21772 70	35 07	6·3	·05014 83	2 38	·05097 37	2 80
1·4	·18137 46	64 29	·21099 40	22 77	6·4	·04938 09	2 25	·05015 44	2 70
1·5	0·17334 51	+ 58 24	0·20403 03	− 12 09	6·5	0·04863 61	+2 20	0·04936 20	+2 53
1·6	·16589 92	52 81	·19694 27	− 3 06	6·6	·04791 32	2 08	·04859 50	2 42
1·7	·15898 25	47 97	·18982 18	+ 4 51	6·7	·04721 11	1 99	·04785 22	2 28
1·8	·15254 65	43 71	·18274 36	10 77	6·8	·04652 89	1 92	·04713 23	2 20
1·9	·14654 83	39 81	·17577 08	15 76	6·9	·04586 59	1 83	·04643 44	2 09
2·0	0·14094 90	+ 36 37	0·16895 36	+ 19 68	7·0	0·04522 12	+1 76	0·04575 74	+1 99
2·1	·13571 41	33 30	·16233 15	22 69	7·1	·04459 41	1 70	·04510 03	1 90
2·2	·13081 27	30 47	·15593 47	24 81	7·2	·04398 40	1 61	·04446 22	1 82
2·3	·12621 66	28 00	·14978 47	26 25	7·3	·04339 01	1 58	·04384 23	1 74
2·4	·12190 09	25 73	·14389 61	27 11	7·4	·04281 19	1 48	·04323 98	1 66
2·5	0·11784 29	+ 23 67	0·13827 76	+ 27 41	7·5	0·04224 86	+1 47	0·04265 39	+1 60
2·6	·11402 20	21 84	·13293 25	27 35	7·6	·04169 99	1 37	·04208 40	1 53
2·7	·11041 98	20 16	·12786 03	26 94	7·7	·04116 50	1 34	·04152 94	1 46
2·8	·10701 95	18 65	·12305 70	26 28	7·8	·04064 35	1 29	·04098 94	1 41
2·9	·10380 59	17 26	·11851 61	25 40	7·9	·04013 49	1 25	·04046 35	1 35
3·0	0·10076 51	+ 16 00	0·11422 89	+ 24 37	8·0	0·03963 88	+1 20	0·03995 11	+1 32
3·1	·09788 45	14 86	·11018 53	23 29	8·1	·03915 47	1 15	·03945 18	1 23
3·2	·09515 27	13 81	·10637 44	22 08	8·2	·03868 21	1 12	·03896 49	1 20
3·3	·09255 92	12 85	·10278 43	20 90	8·3	·03822 07	1 08	·03849 00	1 16
3·4	·09009 44	11 99	·09940 31	19 64	8·4	·03777 01	1 04	·03802 67	1 12
3·5	0·08774 96	+ 11 16	0·09621 84	+ 18 46	8·5	0·03732 99	+1 01	0·03757 46	+1 07
3·6	·08551 66	10 47	·09321 83	17 26	8·6	·03689 98	95	·03713 32	1 04
3·7	·08338 83	9 76	·09039 09	16 15	8·7	·03647 93	96	·03670 22	1 00
3·8	·08135 77	9 14	·08772 50	15 04	8·8	·03606 83	91	·03628 12	96
3·9	·07941 86	8 59	·08520 96	14 00	8·9	·03566 64	86	·03586 98	93
4·0	0·07756 54	+ 8 03	0·08283 43	+ 13 02	9·0	0·03527 32	+ 87	0·03546 77	+ 90
4·1	·07579 26	7 57	·08058 93	12 11	9·1	·03488 86	82	·03507 46	87
4·2	·07409 55	7 12	·07846 55	11 22	9·2	·03451 22	80	·03469 02	86
4·3	·07246 96	6 67	·07645 41	10 44	9·3	·03414 38	77	·03431 43	79
4·4	·07091 05	6 31	·07454 72	9 70	9·4	·03378 31	75	·03394 64	79
4·5	0·06941 45	+ 5 95	0·07273 74	+ 8 98	9·5	0·03342 99	+ 72	0·03358 64	+ 77
4·6	·06797 80	5 60	·07101 76	8 39	9·6	·03308 39	71	·03323 41	73
4·7	·06659 76	5 31	·06938 17	7 77	9·7	·03274 50	69	·03288 91	71
4·8	·06527 03	4 99	·06782 36	7 23	9·8	·03241 30	64	·03255 12	69
4·9	·06399 30	4 77	·06633 79	6 76	9·9	·03208 75	65	·03222 02	68
5·0	0·06276 33	+ 4 48	0·06491 98	+ 6 28	10·0	0·03176 85	+ 63	0·03189 60	+ 65

NOTES ON THE SOLUTION OF THE EQUATION $y'' - xy = f(x)$

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SUMMARY

By suitable approximations and changes of variable, many second-order linear differential equations may be reduced to the Airy Integral form

$$\frac{d^2y}{dx^2} - xy = f(x).$$

In these notes it is shown how this equation may be solved when

$$f(x) = g(x).u + h(x).\frac{du}{dx},$$

with

$$u = a\,\mathrm{Ai}(x) + b\,\mathrm{Bi}(x),$$

and where $g(x)$ and $h(x)$ are expressed as power series. The solution may be expressed in the same form, or as a series of derivatives of u. The solution is also given in the case where $f(x)$ is itself expressed as a power series; in this case it is of the form

$$y = k(x) + l(x).v,$$

where

$$v = u + \pi\mathrm{Gi}(x)$$

in which $k(x)$ and $l(x)$ are expressed as power series. The function $\mathrm{Gi}(x)$ is tabulated in a previous paper (Scorer, **1**). The series in the solution terminates if $g(x)$ and $h(x)$ are polynomials, or if $f(x)$ is a polynomial.

1. It is well known that many problems in mathematics and physics depend on the solution of linear differential equations of the second order. By appropriate choice of dependent or independent variable, or both, such equations may be reduced to the normal form, that is, to a form not containing a term involving the first derivative of the dependent variable. We may therefore, without loss of generality, write the equation to be solved in the form

$$\frac{d^2y}{dx^2} + I(x).y = F(x). \tag{1.1}$$

Over an appropriate range of the independent variable we may in certain cases expand the coefficient $I(x)$ in a power series

$$I(x) = I_0 + I_1.(x-x_0) + I_2.(x-x_0)^2/2! + \dots, \tag{1.2}$$

and an approximation to the solution, sufficient for many purposes, may be found by neglecting terms other than the first one, two, or three. It may be possible to divide the full range of x for which a solution is desired

into a number of intervals, each such that a one-, two-, or three-term expansion (1.2) is an adequate approximation. The solutions valid within the various intervals may then be fitted together at the ends of the intervals, and an approximation obtained covering the whole range of x under consideration.

If we take only the first term of (1.2), the function $I(x)$ is replaced by a constant in each interval of x, that is, by a step-function. In each interval, (1.1) becomes the familiar equation with sine and cosine, or with positive and negative exponentials, as complementary solutions.

If we take intervals in x such that $I(x)$ may be interpolated with second differences only, third differences being negligible, then a quadratic approximation to $I(x)$ is taken, and the equation (1.1) has a complementary solution in terms of Weber functions.

This paper is concerned with cases where intervals of x are chosen so that linear interpolation of $I(x)$ is sufficient, i.e. so that only the terms $I_0+I_1(x-x_0)$ are retained in (1.2). In this case the complementary function in (1.1) is expressed in terms of the Airy Integral $\mathrm{Ai}(\xi)$ and the associated function $\mathrm{Bi}(\xi)$, by writing

$$I_1 x+(I_0-I_1 x_0) = -I_1^{\frac{2}{3}}\xi. \tag{1.3}$$

The equation (1.1) is then converted to the form

$$\frac{d^2y}{d\xi^2}-\xi y = f(\xi). \tag{1.4}$$

This is the equation dealt with below, though x will be used in place of ξ.

2. Using suffixes to denote differentiation with respect to x, the equation to be solved, for certain forms of $f(x)$ is

$$y_2-xy = f(x). \tag{2.1}$$

This suffix notation for differentiation will be applied only to the variables y and u below.

Before dealing with the equation (2.1) it is convenient to develop certain properties of the complementary function. We write

$$\vartheta = \left(\frac{d^2}{dx^2}-x\right) \tag{2.2}$$

for the operator on the left of (2.1), which becomes $\vartheta y = 0$, with the general solution

$$u = a\,\mathrm{Ai}(x)+b\,\mathrm{Bi}(x), \tag{2.3}$$

in which a and b are arbitrary constants, while $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are the Airy Integral functions, as defined by Jeffreys (**2**, pp. 476 and 477) and tabulated in (**3**).

By differentiation it is readily verified that

$$u_{n+2} = xu_n + nu_{n-1} \quad (n \geqslant 0), \tag{2.4}$$

whence u_n can be expressed in the form

$$u_n = p_n(x).u + q_n(x).u_1, \tag{2.5}$$

where $p_n(x)$ and $q_n(x)$ are polynomials of degree not exceeding $\frac{1}{2}n+1$. Thus,

$$\left.\begin{array}{ll} u_2 = xu & u_6 = (x^3+4)u+6xu_1 \\ u_3 = u+xu_1 & u_7 = 9x^2u+(x^3+10)u_1 \\ u_4 = x^2u+2u_1 & u_8 = (x^4+28x)u+12x^2u_1 \\ u_5 = 4xu+x^2u_1 & u_9 = (16x^3+28)u+(x^4+52x)u_1 \end{array}\right\}. \tag{2.6}$$

3. The relation (2.4) may be written as

$$\vartheta u_n = nu_{n-1} \quad (n \geqslant 0), \tag{3.1}$$

which gives an immediate solution to (2.1) when $f(x) = u_{n-1}$, namely $y = u_n/n$. Thus, if

$$f(x) = a_0 u + a_1 u_1 + a_2 u_2 + a_3 u_3 + ..., \tag{3.2}$$

then

$$y = a_0 u_1 + \tfrac{1}{2}a_1 u_2 + \tfrac{1}{3}a_2 u_3 + \tfrac{1}{4}a_3 u_4 + ... \tag{3.3}$$

The form (3.2) for $f(x)$ is less unlikely than at first it appears to be. For if

$$f(x) = g(x).u + h(x).u_1, \tag{3.4}$$

where $g(x)$ and $h(x)$ may be expressed as power series,

$$\left.\begin{array}{l} g(x) = g_0+g_1 x+g_2 x^2+... \\ h(x) = h_0+h_1 x+h_2 x^2+... \end{array}\right\}, \tag{3.5}$$

then, by application of (2.4), repeated if necessary, to eliminate powers of x, we find

$$\left.\begin{array}{ll} u = u & u_1 = u_1 \\ xu = u_2 & xu_1 = u_3-u \\ x^2u = u_4-2u_1 & x^2u_1 = u_5-4u_2 \\ x^3u = u_6-6u_3+2u & x^3u_1 = u_7-9u_4+8u_1 \\ x^4u = u_8-12u_5+20u_2 & x^4u_1 = u_9-16u_6+44u_3-8u \\ x^5u = u_{10}-20u_7+80u_4-40u_1 & x^5u_1 = u_{11}-25u_8+140u_5-140u_2 \end{array}\right\}. \tag{3.6}$$

Thus (3.4) and (3.5) may be expressed in the form (3.2), where

$$\left.\begin{array}{l} a_0 = g_0+2g_3+40g_6+...-h_1-8h_4-280h_7-... \\ a_1 = -2g_2-40g_5-...+h_0+8h_3+280h_6+... \\ a_2 = g_1+20g_4+...-4h_2-140h_5-... \\ a_3 = -6g_3-...+h_1+44h_4+... \end{array}\right\}. \tag{3.7}$$

giving the solution (3.3), which may, if desired, be converted to a form like (3.4) by use of (2.6).

This method of solution is most convenient when $g(x)$ and $h(x)$ are polynomials, or may be treated as such.

4. Next let us suppose that

$$f(x) = b_0+b_1x+b_2x^2+b_3x^3+\dots. \tag{4.1}$$

To solve (2.1) in this case we note that

and, in particular, that

$$\left.\begin{aligned} \vartheta x^n &= n(n-1)x^{n-2}-x^{n+1} \\ \vartheta 1 &= -x \\ \vartheta x &= -x^2 \\ \vartheta x^2 &= -x^3+2 \\ \vartheta x^3 &= -x^4+6x \end{aligned}\right\} \tag{4.2}$$

while it is readily verified that

$$v = \pi\,\mathrm{Gi}(x)+u, \tag{4.3}$$

where

$$\mathrm{Gi}(x) = \frac{1}{\pi}\int\limits_0^\infty \sin(ux+\tfrac{1}{3}u^3)\,du, \tag{4.4}$$

satisfies the relation

$$\vartheta v = -1. \tag{4.5}$$

The equations (4.2) may be rewritten to give single powers on the right, thus

$$\left.\begin{aligned} \vartheta(-v) &= 1 \\ \vartheta(-1) &= x \\ \vartheta(-x) &= x^2 \\ \vartheta(-x^2-2v) &= x^3 \\ \vartheta(-x^3-6) &= x^4 \end{aligned}\right\} \tag{4.6}$$

and so on, the general expression being

$$\vartheta\{-x^n-n(n-1)x^{n-3}-n(n-1)(n-3)(n-4)x^{n-6}-\dots\} = x^{n+1}. \tag{4.7}$$

The series in the brackets terminates if $n+1$ is of the form $3k\pm1$; if $n+1 = 3k$, the series continues to the term in x^2, and is followed by a final term

$$-n(n-1)(n-3)(n-4)(n-6)(n-7)\dots5.4.2.1.v. \tag{4.8}$$

The solution to (2.1) with $f(x)$ given by (4.1) is thus

$$\left.\begin{aligned} y &= u-b_0v-b_1-b_2x-b_3(x^2+2v)-b_4(x^3+6)-\dots \\ &= u-v(b_0+1.2b_3+1.2.4.5b_6+\dots)- \\ &\quad -(b_1+2.3b_4+2.3.5.6b_7+\dots)- \\ &\quad -x(b_2+3.4b_5+3.4.6.7b_8+\dots)- \\ &\quad -x^2(b_3+4.5b_6+4.5.7.8b_9+\dots)-\dots \end{aligned}\right\}. \tag{4.9}$$

5. The solution (4.9) may be written as

$$y = u-cv-d_0-d_1x-d_2x^2-d_3x^3-\dots. \tag{5.1}$$

Then, substituting in (2.1) and using (4.1) and (4.2), and equating coefficients of powers of x,

$$\left.\begin{aligned} 2d_2-b_0-c &= 0 \\ (n+1)(n+2)d_{n+2}-b_n-d_{n-1} &= 0 \end{aligned}\right\}. \tag{5.2}$$

These equations do not suffice for the determination of all the coefficients but if c, d_0, and d_1 have been found, e.g. from their explicit forms in (4.9), then (5.2) may be used to obtain further coefficients in succession.

Again if $f(x)$ is a *polynomial* of degree k we may assume $d_n = 0$ for $n > k+2$, and determine coefficients d_{k+2}, d_{k+1}, d_k, etc., in succession.

6. More generally, by writing (2.1) in the form

$$y = \frac{f(x)}{x}-\frac{y_2}{x} \tag{6.1}$$

and using successive approximations where appropriate, we obtain the formal solution

$$y \sim \frac{1}{x}f(x)-\frac{1}{x}\frac{d^2}{dx^2}\frac{f(x)}{x}+\frac{1}{x}\frac{d^2}{dx^2}\left(\frac{1}{x}\frac{d^2}{dx^2}\frac{f(x)}{x}\right)-\frac{1}{x}\frac{d^2}{dx^2}\left\{\frac{1}{x}\frac{d^2}{dx^2}\left(\frac{1}{x}\frac{d^2}{dx^2}\frac{f(x)}{x}\right)\right\}+\dots. \tag{6.2}$$

Similarly we may write $\qquad y_2 = f(x)+xy, \qquad$ (6.3)

whence the formal particular integral (in a condensed notation)

$$y = \iint f(x)\,(dx)^2+\iint x\iint f(x)\,(dx)^4+\iint x\iint x\iint f(x)\,(dx)^6+\dots. \tag{6.4}$$

It is interesting to note that, if $f(x) = -1/\pi$, equations (6.2) and (6.4) give

$$y \sim -\frac{1}{\pi}\left(\frac{1}{x}-\frac{1.2}{x^4}+\frac{1.2.4.5}{x^7}-\dots\right), \tag{6.5}$$

and $$y = -\frac{1}{\pi}\left(\frac{x^2}{1.2}+\frac{x^5}{1.2.4.5}+\frac{x^8}{1.2.4.5.7.8}+\dots\right), \tag{6.6}$$

which are the expansions for $-\mathrm{Hi}(x) = \mathrm{Gi}(x)-\mathrm{Bi}(x)$ and $\mathrm{Gi}(x)-\frac{1}{3}\mathrm{Bi}(x)$ respectively (see (**1**) (4.3) and (3.8)).

7. These ideas have been much more fully worked out in an unpublished Ph.D. thesis (Mursi (**4**)), where extensive tables of coefficients and further related developments are given. This thesis is inaccessible, having been submitted to Farouk I University, Alexandria, Egypt. Various properties of the function $\mathrm{Ai}(x) = \pi\{\frac{1}{3}\mathrm{Bi}(x)-\mathrm{Gi}(x)\}$ are also discussed; several results given in (**1**) have been obtained independently.

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THE INDUCTION OF ELECTRIC CURRENTS IN A UNIFORM CIRCULAR DISK

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SUMMARY

By regarding a uniform disk, or any symmetrically conducting surface of revolution, as composed of an infinite number of co-axial annular circuits, the determination of the electric currents induced by an external field is reduced to the solution of a Fredholm integral equation whose kernel $K(x, x')$ has infinities of order $\log(x-x')$ when $x = x'$. Two methods of solving this equation are described and illustrated. The time constant for a uniform disk is calculated and agrees approximately with the value estimated by Lamb.

1. Introduction

The induction of electric currents in a conducting circular disk by a varying magnetic field was discussed by Lamb (**1**, **2**) by treating the disk as the limit of a thin spheroidal shell when its axis becomes vanishingly small. In this way he obtained solutions for those cases in which the resistance at distance r from the centre of a disk of radius a is a special function of $(a^2-r^2)^{\frac{1}{2}}$, but he was unable to obtain the solution for the case when the resistance is uniform. He was able, however, to estimate a lower limit for the principal time constant of a uniform disk by regarding it as an intermediate case of two known solutions. As far as the writer knows there has been no further mathematical treatment of the uniform disk problem, though Bruckshaw (**3**) has made an experimental investigation of the resulting induced field, with the object of obtaining results useful in geophysical prospecting.

In the present paper a method suggested to the writer by Professor A. T. Price is developed for treating the problem when the conductivity of the disk and the inducing field have axial symmetry. The disk is regarded as composed of a large number of concentric annular circuits, and the problem thereby reduced to the solution of a Fredholm integral equation. Two methods of solving this equation are described and illustrated with a numerical example. The time constant for a uniform disk is calculated and agrees approximately with the value estimated by Lamb.

2. Derivation of the general equations

A thin circular disk of radius a and uniform surface resistance ρ per unit length is situated in a varying uniform magnetic field H. If the field is oblique to the disk, only the normal component will be effective in

inducing currents in it, and these will be symmetrical about the axis. It is therefore sufficient to treat H as normal to the disk.

The disk will be regarded as composed of a large number of concentric annular circuits, each of inner radius r and outer radius $r+\delta r$. The total resistance of the ring $(r, r+\delta r)$ will be $2\pi r\rho/\delta r$ and the total current circulating in it is $I(r)\,\delta r$, where $I(r)$ is the integrated current density. If $N(r,t)$ is the total induction through the circuit at time t, we have

$$2\pi r\rho I(r) = -\frac{\partial}{\partial t}N(r,t). \tag{1}$$

If $M(r,r')$ is the coefficient of mutual induction between the two rings r and r', $N(r,t)$ is given by

$$N(r,t) = \pi r^2 H + \sum_{r'} I(r')\,\delta r'\, M(r,r'). \tag{2}$$

The summation in (2) covers all the circuits including $r' = r$. If δr is infinitesimal, (1) takes the form:

$$I(r) = -\frac{r}{2\rho}pH - \frac{p}{2r\rho}\int_0^a I(r')\frac{M(r,r')}{\pi}\,dr', \tag{3}$$

where p denotes the operator $(\partial/\partial t)$. If p is treated as a parameter, equation (3) is an integral equation of Fredholm type to determine $I(r,p)$.

3. Extension to surfaces of revolution and to axially symmetric inducing fields

This method can be applied to the more general case of a thin non-uniformly conducting surface of revolution, whose conductivity has axial symmetry, situated in a varying magnetic field which is also symmetrical about the axis. Let the equation of the surface be

$$z = f(r), \tag{4}$$

and let the z-component of the inducing field be $F(z,r)$. Consider the annular circuit with radius r and breadth ds. The resistance of this circuit is $2\pi r\rho(r)/ds$, and the current flowing in it is $I(r)\,ds$. The part of N due to the induced currents will now be given by

$$\int_{r_1}^{r_2} M(r,r')I(r')\,ds' = \int_{r_1}^{r_2} M(r,r')I(r')\{1+f'^2(r)\}^{\frac{1}{2}}\,dr', \tag{5}$$

where r_1, r_2 are the radii of the boundaries of the surface. The part due to the inducing field will be

$$\int_0^r 2\pi r' F(z,r')\,dr' = \pi r^2 H(r,t), \text{ say.} \tag{6}$$

Hence

$$I(r) = -\frac{rpH(r,t)}{2\rho(r)} - \frac{p}{2\pi r\rho(r)} \int\limits_{r_1}^{r_2} I(r')M(r,r')\{1+f'^2(r')\}^{\frac{1}{2}}\,dr', \tag{7}$$

which is a Fredholm integral equation with kernel

$$\{1+f'^2(r')\}^{\frac{1}{2}}M(r,r')/2\pi r\rho(r). \tag{8}$$

Now
$$M(r,r') = 4\pi\sqrt{(rr')}\left\{\left(\frac{2}{c}-c\right)K(c)-\frac{2}{c}E(c)\right\}, \tag{9}$$

where
$$c^2 = 4rr'\{(r+r')^2+h^2\}^{-1},$$

h is the normal distance between the planes of the circuits, and $K(c)$, $E(c)$ are complete elliptic integrals of the first and second kinds respectively. The coefficient of mutual induction will be finite everywhere except when $c = 1$. This is the case when $r = r'$, $h = 0$. In this case $E(c)$ behaves like $\log(r-r')$ and $M(r,r')$ reduces to the coefficient of self-induction; we can then use the well-known approximate value for the self-induction of an annular ring of inner and outer radii r, r' respectively, namely

$$M(r,r') = 4\pi r[\log\{8r/(r-r')\}-\tfrac{7}{4}]. \tag{10}$$

Thus the kernel of the integral equation (7) is finite except at $r = r'$, $h = 0$, where it becomes infinite like $\log(r-r')$.

4. The uniform disk

The solution of the integral equation (3) will now be considered; the general equation (7) can be treated on parallel lines. If we write $x = r/a$, so that $M(r,r') = aM(x,x')$, and also write $I(x)$ for $I(r)$, equation (3) takes the form

$$I(x) = \lambda xH + \frac{\lambda}{\pi x}\int\limits_0^1 I(x')M(x,x')\,dx', \tag{11}$$

where
$$\lambda = -ap/2\rho. \tag{12}$$

Equation (11) can be transformed to another with symmetrical kernel, namely

$$\psi(x) = \lambda Hx^{\frac{3}{2}} + 4\lambda\int\limits_0^1 \psi(x')K(x,x')\,dx', \tag{13}$$

where

$$\begin{gathered}\psi(x) = x^{\frac{1}{2}}I(x),\\ K(x,x') = M(x,x')/\{4\pi\sqrt{(xx')}\}.\end{gathered} \tag{14}$$

The new kernel $K(x,x')$ will still have an infinity of order $\log(x-x')$ when $x = x'$.

One method of solving (13) approximately is to regard the disk as composed of, say, ten circuits each of breadth $(a/10)$ and with mean radii $0{\cdot}05a$, $0{\cdot}15a$, $0{\cdot}25a$,..., and $0{\cdot}95a$ respectively. When calculating the

coefficient of mutual induction between any two of these circuits they are treated as line circuits. The expression (10) was not, however, used to calculate the coefficient of self-induction for the above circuits, because the breadth of each circuit (especially those nearest the centre of the disk) although small, cannot be neglected in comparison with the mean radius of the circuit. Instead each ring of mean radius r was further subdivided into ten annular rings, each of breadth $0{\cdot}01a$, and of mean radii r_1, r_2,..., and r_{10}, where

$$r_s = r+(0{\cdot}01s-0{\cdot}055)a. \tag{15}$$

If $M(r,r_1)$, $M(r,r_2)$,..., $M(r,r_{10})$ are the coefficients of mutual induction between the ten new rings and the original ring, its coefficient of self-induction is approximately

$$\tfrac{1}{10}\sum_{s=1}^{10} M(r,r_s). \tag{16}$$

Using the values for $\log\{M(x,x')/4\pi\sqrt{(xx')}\}$ given by Maxwell (**4**), the values of $K(x,x')$ are readily obtained from (14) and (16).

5. The solution of the integral equation by the method of 'continued substitution'

In this method, which is a special case of a general method due to Price (**5**) for any non-uniformly conducting sheet, a first approximation $\psi_1(x)$ is obtained for $\psi(x)$ by neglecting the second term on the right-hand side of (13). Thus we have

$$\psi_1(x) = \lambda H x^{\frac{3}{2}} = \lambda H\phi_0(x). \tag{17}$$

This approximation corresponds to neglecting the self-induction of the disk. The second approximation is obtained by substituting $\psi_1(x')$ for $\psi(x')$ in (13). We get

$$\psi_2(x) = \lambda H\phi_0(x)+\lambda^2 H\phi_1(x), \tag{18}$$

where

$$\phi_1(x) = 4\int_0^1 \phi_0(x')K(x,x')\,dx'. \tag{19}$$

By continuing this process, the nth approximation will be given by

$$\psi_n(x) = \lambda H\sum_{m=1}^{n} \phi_{m-1}(x)\lambda^{m-1}, \tag{20}$$

where

$$\phi_m(x) = 4\int_0^1 \phi_{m-1}(x')K(x,x')\,dx'. \tag{21}$$

Hille and Tamarkin (**6**) have shown that the equation

$$\phi(x) = f(x)+\lambda\int_a^b K(x,x')\phi(x')\,dx', \tag{22}$$

where $K(x,x')$ is symmetric and has discontinuities when $x = x'$, admits

a convergent solution by the method of continued substitution provided that

(i) $f(x)$ is integrable (in the sense of Lebesgue);

(ii) $\dfrac{1}{b-a}\displaystyle\int\limits_a^b K(x,x')\,dx \leqslant \left|\dfrac{1}{\lambda}\right|.$

Hence, since $x^{\frac{1}{2}}$ is integrable, the infinite series obtained by making n tend to infinity in (20) will represent the solution of (13) provided that

$$|\lambda| \leqslant \left\{4\int\limits_0^1 K(x,x')\,dx\right\}^{-1}. \tag{23}$$

It follows that $$I(x) = \psi(x)x^{-\frac{1}{2}} = \lambda H x^{-\frac{1}{2}} \sum_{m=0}^{\infty} \phi_m(x)\lambda^m. \tag{24}$$

The functions $\phi_n(x)$ are evaluated at the points $x = 0{\cdot}05,\ 0{\cdot}15,\ldots,$ and $0{\cdot}95$ using their recurrence formula (21), and the values $x^{-\frac{1}{2}}\phi_n(x)$ for the above values of x and $n = 1, 2,\ldots,$ and 8, are tabulated in Table I.

TABLE I. *Values of* $x^{-\frac{1}{2}}\phi_n(x)$

n \ x	0·05	0·15	0·25	0·35	0·45	0·55	0·65	0·75	0·85	0·95
0	0·0500	0·15000	0·25000	0·35000	0·45000	0·55000	0·65000	0·75000	0·85000	0·95000
1	0·3000	0·92240	1·5171	2·0527	2·5641	2·9958	3·3199	3·5078	3·4889	3·1616
2	1·5640	4·8115	7·8373	10·573	12·767	14·515	15·517	15·679	14·771	12·647
3	7·5891	23·324	37·771	50·578	60·189	65·736	70·678	69·955	64·492	54·348
4	35·405	108·65	175·29	233·57	275·58	305·06	318·33	312·53	285·83	239·67
5	162·28	497·66	801·51	1065·4	1253·6	1385·8	1435·5	1402·5	1288·0	1069·5
6	737·31	2260·3	3637·1	4829·1	5637·5	6261·4	5468·9	6306·5	5737·3	4797·7
7	3335·3	10222	16442	21817	25611	28238	29143	28384	25803	21570
8	15054	46133	74185	98408	115470	127260	131240	127800	116150	97077

This table actually contains the coefficients of the powers of λ in the series representing the solution of the integral equation, and hence it can be used in solving any particular problem with any special value of the operator λ. In case of periodic external fields, if we only require the steady state solution, we may write H as the real part of $He^{i\alpha t}$; the operator $\partial/\partial t$ can then be replaced by $i\alpha$.

As an illustration, a numerical case is considered which is of interest in discussing the effects of a large area of sea water in geomagnetism. A disk of radius 1,000 km., thickness 5 km., and specific conductivity 4×10^{-11} e.m.u. is situated in a uniform field normal to it and having simple harmonic variations of period one day, say $H = H\cos\alpha t$. These values make $\lambda = -0{\cdot}07268i$. The amplitude and phase of the current density, which

is represented by the real part of the right-hand side of (24) are calculated and tabulated in Table II.

TABLE II

x	0·05	0·15	0·25	0·35	0·45	0·55	0·65	0·75	0·85	0·95
$R(x)$	0·0034	0·0102	0·0170	0·0238	0·0308	0·0377	0·447	0·0519	0·0591	0·0664
$\alpha(x)$	65·8°	65·1°	65·5°	66·4°	67·0°	67·9°	69·4°	71·1°	73·5°	76·7°

The amplitude and phase of the induced current density
$I(x) = -HR(x)\cos\{\alpha t+\alpha(x)\}$ e.m.u.

The above table shows that the amplitude increases steadily as the distance from the centre increases, till it attains its maximum at the boundary. On the other hand, there is little change in the phase of the induced current density. If the self-inductance of the disk be neglected, the induced current density will be given simply by

$$I(x) = -x\lambda H, \tag{25}$$

so that $R(x) = x$ and $\alpha(x) = \pi/2$.

6. Fredholm solution of the integral equation

Fredholm (*vide* (**7**)) has shown that the unique solution of the equation

$$\psi(x) = f(x)+\lambda \int\limits_a^b \psi(x')K(x,x')\,dx', \tag{26}$$

when the kernel is continuous and finite for all x, x' in the interval (a,b), is given by

$$\psi(x) = f(x)+\frac{1}{\nabla(\lambda)} \int\limits_a^b f(x')\nabla(x,x',\lambda)\,dx', \tag{27}$$

where

$$\left.\begin{aligned}
&\nabla(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} a_n \lambda^n, \qquad \nabla(x,x',\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} b_n(x,x')\lambda^{n+1},\\
&a_n = \int\limits_a^b \cdots \int\limits_a^b D_n\,dx_1\,dx_2 \ldots dx_n,\\
&b_n(x,x') = \int\limits_a^b \cdots \int\limits_a^b D_n(x,x')\,dx_1\,dx_2 \ldots dx_n,\\
&D_n \text{ denotes the determinant } \Delta\{K(x_r,x_s)\},\\
&\qquad r = 1,2,\ldots,n;\ s = 1,2,\ldots,n,\\
&\text{and}\quad D_n(x,x') = \begin{vmatrix} K(x,x') & K(x,x_1) & . & . & . & K(x,x_n) \\ K(x_1,x') & K(x_1,x_1) & . & . & . & K(x_1,x_n) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ K(x_n,x') & K(x_n,x_1) & . & . & . & K(x_n,x_n) \end{vmatrix}.
\end{aligned}\right\} \tag{28}$$

This solution is not immediately applicable to (13) because the kernel $K(x,x')$ has an infinity at $x = x'$. Bôcher (**8**) introduced the modified functions $\overline{\nabla}(\lambda)$, $\overline{\nabla}(x,x',\lambda)$. These are obtained from the solving functions $\nabla(\lambda)$, $\nabla(x,x',\lambda)$ by putting $K(x_r,x_r) = 0$ $(r = 1,2,...,n)$ in the principal diagonal of D_n and $D_n(x,x')$. The corresponding values for a_n, $b_n(x,x')$, D_n and $D_n(x,x')$ will be denoted by $\bar{a}_n$, $\bar{b}_n(x,x')$, $\bar{D}_n$ and $\bar{D}_n(x,x')$ respectively. Bôcher then proved that for any symmetrical kernel whose discontinuities are regularly distributed, the modified functions do not differ from the solving functions except by a constant exponential factor e^c, i.e.

$$\left.\begin{aligned} \nabla(\lambda) &= \exp(c)\overline{\nabla}(\lambda), \\ \nabla(x,x',\lambda) &= \exp(c)\overline{\nabla}(x,x',\lambda). \end{aligned}\right\} \tag{29}$$

Hence for such kernels the solution of (25) is not affected if the modified functions are used instead of the solving functions. Also, the characteristic numbers of the kernel are the zeroes of either $\nabla(\lambda)$ or $\overline{\nabla}(\lambda)$.

Hilbert (**9**) has extended this theory to symmetric kernels having infinities on the line $x' = x$ when a positive number $\beta < \frac{1}{2}$ can be found such that

$$\lim_{x'\to x} (x-x')^{\beta} K(x,x') = 0. \tag{30}$$

In this case he proved that Fredholm's solution will be valid and unique provided that instead of the solving functions the modified functions are used. With the help of the latter functions all infinities are removed. The condition (30) is satisfied by equation (13), since for *any* positive number β

$$\lim_{x'\to x} (x-x')^{\beta} \log(x-x') = 0.$$

As far as the writer knows no simpler forms for $\bar{a}_n$, $\bar{b}_n(x,x')$ have been found. So, since this is necessary for any practical application, the following forms are obtained for any symmetric kernel.

From the definition we have by expanding $\bar{D}_n$

$$\bar{a}_n = (n-1)! \sum_{r=2}^{n} \frac{(-1)^{r-1}}{(n-r)!} I_r \bar{a}_{n-r}, \quad n > 1, \quad \bar{a}_0 = 1, \quad \bar{a}_1 = 0, \tag{31}$$

where

$$\left.\begin{aligned} I_2 &= \int_a^b\!\!\int_a^b K^2(x_1,x_2)\, dx_1 dx_2, \\ I_r &= \int_a^b ... \int_a^b K(x_1,x_2)K(x_2,x_3)...K(x_{r-1},x_r)K(x_r,x_1)\, dx_1 dx_2...dx_r. \end{aligned}\right\} \tag{32}$$

Similarly $$\bar{b}_n(x,x') = n!\sum_{r=0}^{n} \frac{(-1)^r}{(n-r)!} I_r(x,x')\bar{a}_{n-r}, \tag{33}$$

where $$I_0(x,x') = K(x,x'),$$
$$I_r(x,x') = \int_a^b I_{r-1}(x,\xi)I_0(\xi,x')\,d\xi \quad (r \geqslant 1). \tag{34}$$

The following two useful relations follow immediately:

$$\left.\begin{aligned} I_{r+2} &= \int_a^b\int_a^b K(x,x')I_r(x,x')\,dxdx', \\ \bar{a}_{r+2} &= -(r+1)\int_a^b\int_a^b K(x,x')\bar{b}_n(x,x')\,dxdx' \end{aligned}\right\}. \tag{35}$$

All functions of two variables needed for our special case have been evaluated for the hundred points (x,x'), where $x = 0{\cdot}05, 0{\cdot}15,..., 0{\cdot}95$ and $x' = 0{\cdot}05, 0{\cdot}15,..., 0{\cdot}95$. The series $\bar{\nabla}(\lambda)$ and $\bar{\nabla}(x,x',\lambda)$ are evaluated as far as the term λ^4. Starting with $I_0(x,x') = K(x,x')$, the functions $I_n(x,x')$, $n = 1$, 2, and 3 are evaluated by carrying the integration in (34). The integrals I_2, I_3, and I_4 are then evaluated using (35). Their values are found to be

$$I_2 = 1{\cdot}8161, \qquad I_3 = 1{\cdot}6404, \qquad I_4 = 1{\cdot}7018. \tag{36}$$

The coefficients $\bar{a}_n$ were then evaluated, giving

$$\bar{\nabla}(\lambda) = 1-0{\cdot}9080\lambda^2-0{\cdot}5468\lambda^3-0{\cdot}0132\lambda^4.... \tag{37}$$

It will also be seen that

$$\begin{aligned} \bar{b}_0(x,x') &= I_0(x,x'), & \bar{b}_2 &= -I_2\,I_0(x,x')+2I_2(x,x'), \\ \bar{b}_1(x,x') &= I_1(x,x'), & \bar{b}_3 &= -2I_3\,I_0(x,x')+3I_2\,I_1(x,x')-6I_3(x,x'), \end{aligned} \tag{38}$$

and thus the first four terms in $\bar{\nabla}(x,x',\lambda)$ can be obtained readily. The solution of (11) can now be obtained in the form

$$I(x) = \lambda Hx + \frac{4\lambda^2 H}{x^{\frac{1}{2}}\bar{\nabla}(\lambda)}\int_0^1 x'^{\frac{3}{2}}\bar{\nabla}(x,x',\lambda)\,dx'. \tag{39}$$

The special numerical case already discussed in section 5 was also calculated by the present method. The amplitude and phase of the current density are obtained at the same points as before. The results, given in Table III, agree well with those in Table II.

TABLE III

x	0·05	0·15	0·25	0·35	0·45	0·55	0·65	0·75	0·85	0·95
$R(x)$	0·0034	0·0102	0·0170	0·0239	0·0308	0·0377	0·0447	0·0519	0·0591	0·0664
$\alpha(x)$	66·3°	65·3°	65·5°	66·0°	66·9°	67·9°	69·4°	71·1°	73·5°	76·6°

The amplitude and phase of the induced current density calculated by Fredholm's method.

7. The free currents in the disk

If a symmetrical set of currents is excited in the disk and left to decay, there being no inducing field, equation (13) will be homogeneous. Assuming the current density everywhere to be decaying exponentially like $e^{-t/\tau}$, the operator $\partial/\partial t$ may be replaced by $-\tau^{-1}$ and λ is then equal to $2a/\tau\rho$. The characteristic numbers of the kernel thus give the values of the time constant τ.

Lamb's discussion of this problem led to the fixing of a lower limit to the principal time constant, namely $\tau = 2{\cdot}26a/\rho$. This means an upper limit to $\lambda = 0{\cdot}885$. This value should correspond to the smallest zero of $\nabla(\lambda)$ which is given approximately by (37). Using the five terms of $\nabla(\lambda)$ given in (37), we obtain $\lambda = 0{\cdot}850$, which gives $\tau = 2{\cdot}34a/\rho$. This value, which is probably in error in the last figure, is slightly larger than Lamb's estimate which was a lower limit; the agreement is satisfactory and affords some check on the calculations.†

It is of interest in geomagnetism to estimate the rate at which free currents decay in the oceans. If an ocean is regarded as a large uniform disk with radius a and depth d, the time constant, i.e. the time required for the current density to reduce to a fraction $1/e$ of its initial value, can be calculated by using the above value of τ. This is calculated for the values of a and d shown in Table IV, taking the conductivity of sea water to be $K = 4\times10^{-11}$ e.m.u.

TABLE IV

d \ c	1	2	3	4	5
1	0·26				
2	0·52	1·04			
3	0·78	1·56	2·34		
4	1·04	2·08	3·12	4·16	
5	1·30	2·60	3·90	5·20	6·50

The time constant (in hours) for a circular ocean of radius $a = 1000c$ km. and depth d km.

In conclusion I wish to thank Professor A. T. Price of the Imperial College, London, for suggesting this investigation and for his interest and help in its development.

† One of Lamb's actual problems will be discussed by this method in a following paper.

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